Exercise 26

Find the line through $(3, 1, -2)$ that intersects and is perpendicular to the line $x = -1 + t$, $y = -2 + t$, $z = -1 + t$ [HINT: If (x_0, y_0, z_0) is the point of intersection, find its coordinates.]

Solution

The equation for the line we're trying to find is

$$\mathbf{r}(t) = \mathbf{m}t + \mathbf{b}.$$

We require that the direction vector $\mathbf{m}$ be orthogonal to the direction vector of the given line.

$$(-1 + t, -2 + t, -1 + t) = (t, t, t) + (-1, -2, -1) = (1, 1, 1)t + (-1, -2, -1)$$

In other words, the dot product of $\mathbf{m}$ and $(1, 1, 1)$ is zero.

$$\mathbf{m} \cdot (1, 1, 1) = 0$$

$$(m_x, m_y, m_z) \cdot (1, 1, 1) = 0$$

$$m_x + m_y + m_z = 0 \tag{1}$$

The sum of the components must be zero. Make it so that the line goes through $(3, 1, -2)$ at $t = 0$.

$$\mathbf{r}(0) = (3, 1, -2) = \mathbf{b}$$

As a result,

$$\begin{aligned} \mathbf{r}(t) &= (m_x, m_y, m_z)t + (3, 1, -2) \\ &= (m_x t, m_y t, m_z t) + (3, 1, -2) \\ &= (m_x t + 3, m_y t + 1, m_z t - 2). \end{aligned}$$

Also, make it so that this line and the given line intersect at $t = t_0$.

$$\mathbf{r}(t_0) = (m_x t_0 + 3, m_y t_0 + 1, m_z t_0 - 2) = (-1 + t_0, -2 + t_0, -1 + t_0)$$

Match the components to obtain a system of equations.

$$\begin{aligned} m_x t_0 + 3 &= -1 + t_0 \\ m_y t_0 + 1 &= -2 + t_0 \\ m_z t_0 - 2 &= -1 + t_0 \end{aligned}$$

Together with equation (1) there are four equations and four unknowns. Solving it yields

$$m_x = -1 \quad \text{and} \quad m_y = -\frac{1}{2} \quad \text{and} \quad m_z = \frac{3}{2} \quad \text{and} \quad t_0 = 2.$$

Therefore, an equation for the desired line is

$$\mathbf{r}(t) = \left(-1, -\frac{1}{2}, \frac{3}{2}\right) t + (3, 1, -2).$$