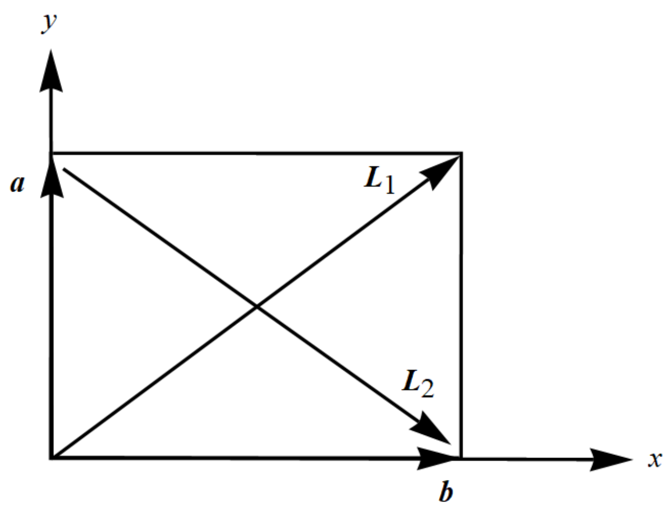
Exercise 39

Using vectors, show that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

Solution

Let the two adjacent sides of the rectangle be represented by vectors $\mathbf{a}$ and $\mathbf{b}$. Also, let the diagonals be represented by vectors $\mathbf{L}_1$ and $\mathbf{L}_2$.



From the figure,

$$\mathbf{a} = (0, a)$$
$$\mathbf{b} = (b, 0)$$

and

$$\mathbf{L}_1 = \mathbf{b} + \mathbf{a} = (b, a)$$
$$\mathbf{L}_2 = \mathbf{b} - \mathbf{a} = (b, -a).$$

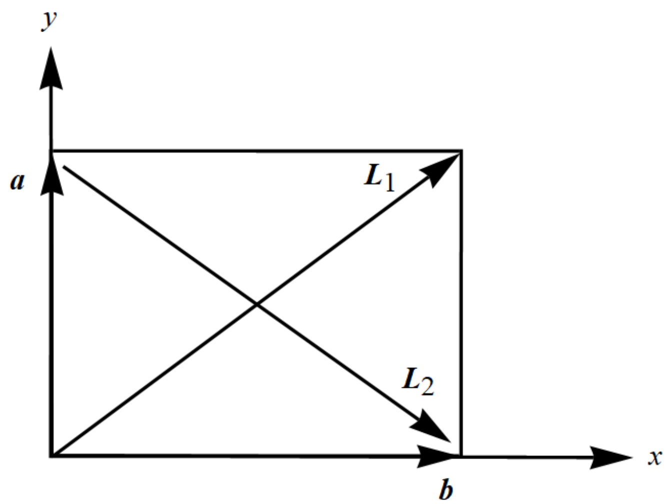
Suppose that the diagonals are perpendicular. Then the dot product of $\mathbf{L}_1$ and $\mathbf{L}_2$ must be zero.

$$\mathbf{L}_1 \cdot \mathbf{L}_2 = 0$$

$$(b, a) \cdot (b, -a) = 0$$
$$b^2 - a^2 = 0$$

As a result, $a = b$, which means all sides of the rectangle have the same length.

Suppose now that the rectangle is a square, and all sides of the rectangle are the same: $a = b$.



From the figure,

$$\begin{aligned}\mathbf{a} &= (0, a)\\ \mathbf{b} &= (a, 0)\end{aligned}$$

and

$$\begin{aligned}\mathbf{L}_1 &= \mathbf{b} + \mathbf{a} = (a, a)\\ \mathbf{L}_2 &= \mathbf{b} - \mathbf{a} = (a, -a).\end{aligned}$$

Calculate the dot product of $\mathbf{L}_1$ and $\mathbf{L}_2$ to determine the angle between them.

$$\mathbf{L}_1 \cdot \mathbf{L}_2 = \|\mathbf{L}_1\|\|\mathbf{L}_2\| \cos\theta$$

Solve for $\cos\theta$.

$$\begin{aligned}\cos\theta &= \frac{\mathbf{L}_1 \cdot \mathbf{L}_2}{\|\mathbf{L}_1\|\|\mathbf{L}_2\|}\\ &= \frac{(a, a) \cdot (a, -a)}{\sqrt{a^2 + a^2}\sqrt{a^2 + (-a)^2}}\\ &= \frac{a^2 - a^2}{\sqrt{2a^2}\sqrt{2a^2}}\\ &= 0\end{aligned}$$

The angle between the diagonals is then

$$\theta = \cos^{-1} 0 = \frac{\pi}{2},$$

which means they are perpendicular. Therefore, the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.