Exercise 16

Find an equation for the plane that passes through

(a) $(0, 0, 0)$, $(2, 0, -1)$, and $(0, 4, -3)$.

(b) $(1, 2, 0)$, $(0, 1, -2)$, and $(4, 0, 1)$.

(c) $(2, -1, 3)$, $(0, 0, 5)$, and $(5, 7, -1)$.

Solution

A particular plane is specified by a normal vector $\mathbf{n}$ and a point with position vector $\mathbf{r}_0$ that lies in it. The equation for this plane comes from the fact that the dot product of $\mathbf{n}$ with any vector in the plane is zero.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Part (a)

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(0, 0, 0)$ to $(2, 0, -1)$ and $(0, 4, -3)$, respectively.

$$\mathbf{r}_1 = (2, 0, -1) - (0, 0, 0) = (2, 0, -1)$$
$$\mathbf{r}_2 = (0, 4, -3) - (0, 0, 0) = (0, 4, -3)$$

Take the cross product of these two to obtain a vector normal to the plane that they reside in.

$$\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 0 & -1 \\ 0 & 4 & -3 \end{vmatrix} = (0 + 4)\hat{\mathbf{x}} - (-6 - 0)\hat{\mathbf{y}} + (8 - 0)\hat{\mathbf{z}} = 4\hat{\mathbf{x}} + 6\hat{\mathbf{y}} + 8\hat{\mathbf{z}} = (4, 6, 8)$$

Therefore, the equation for the plane is

$$(4, 6, 8) \cdot (x - 0, y - 0, z - 0) = 0$$

$$4(x - 0) + 6(y - 0) + 8(z - 0) = 0$$

$$4x + 6y + 8z = 0$$

$$2x + 3y + 4z = 0.$$

Part (b)

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(1, 2, 0)$ to $(0, 1, -2)$ and $(4, 0, 1)$, respectively.

$$\mathbf{r}_1 = (0, 1, -2) - (1, 2, 0) = (-1, -1, -2)$$
$$\mathbf{r}_2 = (4, 0, 1) - (1, 2, 0) = (3, -2, 1)$$

Take the cross product of these two to obtain a vector normal to the plane that they reside in.

$$\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (-1 - 4)\hat{\mathbf{x}} - (-1 + 6)\hat{\mathbf{y}} + (2 + 3)\hat{\mathbf{z}} = -5\hat{\mathbf{x}} - 5\hat{\mathbf{y}} + 5\hat{\mathbf{z}} = (-5, -5, 5)$$

Therefore, the equation for the plane is

$$(-5, -5, 5) \cdot (x - 1, y - 2, z - 0) = 0$$

$$-5(x - 1) - 5(y - 2) + 5(z - 0) = 0$$

$$-5x + 5 - 5y + 10 + 5z = 0$$

$$-5x - 5y + 5z = -15$$

$$x + y - z = 3.$$

Part (c)

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(0, 0, 5)$ to $(2, -1, 3)$ and $(5, 7, -1)$, respectively.

$$\mathbf{r}_1 = (2, -1, 3) - (0, 0, 5) = (2, -1, -2)$$
$$\mathbf{r}_2 = (5, 7, -1) - (0, 0, 5) = (5, 7, -6)$$

Take the cross product of these two to obtain a vector normal to the plane that they reside in.

$$\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & -1 & -2 \\ 5 & 7 & -6 \end{vmatrix} = (6 + 14)\hat{\mathbf{x}} - (-12 + 10)\hat{\mathbf{y}} + (14 + 5)\hat{\mathbf{z}} = 20\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 19\hat{\mathbf{z}} = (20, 2, 19)$$

Therefore, the equation for the plane is

$$(20, 2, 19) \cdot (x - 0, y - 0, z - 5) = 0$$

$$20(x - 0) + 2(y - 0) + 19(z - 5) = 0$$

$$20x + 2y + 19z - 95 = 0$$

$$20x + 2y + 19z = 95.$$