Exercise 23

(a) Prove the two triple-vector-product identities

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

and

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

(b) Prove $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ if and only if $(\mathbf{u}\times\mathbf{w})\times\mathbf{v} = \mathbf{0}$.

(c) Also prove that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} + (\mathbf{v}\times\mathbf{w})\times\mathbf{u} + (\mathbf{w}\times\mathbf{u})\times\mathbf{v} = \mathbf{0}$ (called the ***Jacobi identity***).

Solution

Part (a)

Let $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ and $\mathbf{c} = (c_x, c_y, c_z)$.

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \times\mathbf{c} \\
&= [(a_yb_z - a_zb_y)\hat{\mathbf{x}} - (a_xb_z - a_zb_x)\hat{\mathbf{y}} + (a_xb_y - a_yb_x)\hat{\mathbf{z}}]\times\mathbf{c} \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_yb_z - a_zb_y & a_zb_x - a_xb_z & a_xb_y - a_yb_x \\ c_x & c_y & c_z \end{vmatrix} \\
&= [(a_zb_x - a_xb_z)c_z - (a_xb_y - a_yb_x)c_y]\hat{\mathbf{x}} - [(a_yb_z - a_zb_y)c_z - (a_xb_y - a_yb_x)c_x]\hat{\mathbf{y}} \\
&\qquad + [(a_yb_z - a_zb_y)c_y - (a_zb_x - a_xb_z)c_x]\hat{\mathbf{z}} \\
&= (b_xa_zc_z - a_xb_zc_z - a_xb_yc_y + b_xa_yc_y)\hat{\mathbf{x}} - (a_yb_zc_z - b_ya_zc_z - b_ya_xc_x + a_yb_xc_x)\hat{\mathbf{y}} \\
&\qquad + (b_za_yc_y - a_zb_yc_y - a_zb_xc_x + b_za_xc_x)\hat{\mathbf{z}} \\
&= (b_xa_xc_x + b_xa_yc_y + b_xa_zc_z - a_xb_xc_x - a_xb_yc_y - a_xb_zc_z)\hat{\mathbf{x}} \\
&\qquad + (b_ya_xc_x + b_ya_yc_y + b_ya_zc_z - a_yb_xc_x - a_yb_yc_y - a_yb_zc_z)\hat{\mathbf{y}} \\
&\qquad + (b_za_xc_x + b_za_yc_y + b_za_zc_z - a_zb_xc_x - a_zb_yc_y - a_zb_zc_z)\hat{\mathbf{z}} \\
&= [b_x(a_xc_x + a_yc_y + a_zc_z) - a_x(b_xc_x + b_yc_y + b_zc_z)]\hat{\mathbf{x}} \\
&\qquad + [b_y(a_xc_x + a_yc_y + a_zc_z) - a_y(b_xc_x + b_yc_y + b_zc_z)]\hat{\mathbf{y}} \\
&\qquad + [b_z(a_xc_x + a_yc_y + a_zc_z) - a_z(b_xc_x + b_yc_y + b_zc_z)]\hat{\mathbf{z}} \\
&= [b_x(\mathbf{a}\cdot\mathbf{c}) - a_x(\mathbf{b}\cdot\mathbf{c})]\hat{\mathbf{x}} + [b_y(\mathbf{a}\cdot\mathbf{c}) - a_y(\mathbf{b}\cdot\mathbf{c})]\hat{\mathbf{y}} + [b_z(\mathbf{a}\cdot\mathbf{c}) - a_z(\mathbf{b}\cdot\mathbf{c})]\hat{\mathbf{z}} \\
&= (b_x(\mathbf{a}\cdot\mathbf{c}) - a_x(\mathbf{b}\cdot\mathbf{c}), b_y(\mathbf{a}\cdot\mathbf{c}) - a_y(\mathbf{b}\cdot\mathbf{c}), b_z(\mathbf{a}\cdot\mathbf{c}) - a_z(\mathbf{b}\cdot\mathbf{c})) \\
&= (b_x(\mathbf{a}\cdot\mathbf{c}), b_y(\mathbf{a}\cdot\mathbf{c}), b_z(\mathbf{a}\cdot\mathbf{c})) - (a_x(\mathbf{b}\cdot\mathbf{c}), a_y(\mathbf{b}\cdot\mathbf{c}), a_z(\mathbf{b}\cdot\mathbf{c})) \\
&= (\mathbf{a}\cdot\mathbf{c})(b_x, b_y, b_z) - (\mathbf{b}\cdot\mathbf{c})(a_x, a_y, a_z) \\
&= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}
\end{aligned}$$

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \\
&= \mathbf{a} \times [(b_y c_z - b_z c_y)\hat{\mathbf{x}} - (b_x c_z - b_z c_x)\hat{\mathbf{y}} + (b_x c_y - b_y c_x)\hat{\mathbf{z}}] \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_y c_z - b_z c_y & b_z c_x - b_x c_z & b_x c_y - b_y c_x \end{vmatrix} \\
&= [a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z)]\hat{\mathbf{x}} - [a_x(b_x c_y - b_y c_x) - a_z(b_y c_z - b_z c_y)]\hat{\mathbf{y}} \\
&\qquad + [a_x(b_z c_x - b_x c_z) - a_y(b_y c_z - b_z c_y)]\hat{\mathbf{z}} \\
&= (b_x a_y c_y - c_x a_y b_y - c_x a_z b_z + b_x a_z c_z)\hat{\mathbf{x}} + (b_y a_x c_x + b_y a_z c_z - c_y a_x b_x - c_y a_z b_z)\hat{\mathbf{y}} \\
&\qquad + (b_z a_x c_x - c_z a_x b_x - c_z a_y b_y + b_z a_y c_y)\hat{\mathbf{z}} \\
&= (b_x a_x c_x + b_x a_y c_y + b_x a_z c_z - c_x a_x b_x - c_x a_y b_y - c_x a_z b_z)\hat{\mathbf{x}} \\
&\qquad + (b_y a_x c_x + b_y a_y c_y + b_y a_z c_z - c_y a_x b_x - c_y a_y b_y - c_y a_z b_z)\hat{\mathbf{y}} \\
&\qquad + (b_z a_x c_x + b_z a_y c_y + b_z a_z c_z - c_z a_x b_x - c_z a_y b_y - c_z a_z b_z)\hat{\mathbf{z}} \\
&= [b_x(a_x c_x + a_y c_y + a_z c_z) - c_x(a_x b_x + a_y b_y + a_z b_z)]\hat{\mathbf{x}} \\
&\qquad + [b_y(a_x c_x + a_y c_y + a_z c_z) - c_y(a_x b_x + a_y c b_y + a_z b_z)]\hat{\mathbf{y}} \\
&\qquad + [b_z(a_x c_x + a_y c_y + a_z c_z) - c_z(a_x b_x + a_y b_y + a_z b_z)]\hat{\mathbf{z}} \\
&= [b_x(\mathbf{a} \cdot \mathbf{c}) - c_x(\mathbf{a} \cdot \mathbf{b})]\hat{\mathbf{x}} + [b_y(\mathbf{a} \cdot \mathbf{c}) - c_y(\mathbf{a} \cdot \mathbf{b})]\hat{\mathbf{y}} + [b_z(\mathbf{a} \cdot \mathbf{c}) - c_z(\mathbf{a} \cdot \mathbf{b})]\hat{\mathbf{z}} \\
&= (b_x(\mathbf{a} \cdot \mathbf{c}) - c_x(\mathbf{a} \cdot \mathbf{b}), b_y(\mathbf{a} \cdot \mathbf{c}) - c_y(\mathbf{a} \cdot \mathbf{b}), b_z(\mathbf{a} \cdot \mathbf{c}) - c_z(\mathbf{a} \cdot \mathbf{b})) \\
&= (b_x(\mathbf{a} \cdot \mathbf{c}), b_y(\mathbf{a} \cdot \mathbf{c}), b_z(\mathbf{a} \cdot \mathbf{c})) - (c_x(\mathbf{a} \cdot \mathbf{b}), c_y(\mathbf{a} \cdot \mathbf{b}), c_z(\mathbf{a} \cdot \mathbf{b})) \\
&= (\mathbf{a} \cdot \mathbf{c})(b_x, b_y, b_z) - (\mathbf{a} \cdot \mathbf{b})(c_x, c_y, c_z) \\
&= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
\end{aligned}$$

Part (b)

Suppose that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

The aim is to show that $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}$. Use the results from part (a) to write the cross products as dot products.

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Cancel $(\mathbf{u} \cdot \mathbf{w})\mathbf{v}$ from both sides.

$$-(\mathbf{v} \cdot \mathbf{w})\mathbf{u} = -(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Bring both terms to the right side.

$$(\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = \mathbf{0}$$

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = \mathbf{0}$$

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u} = \mathbf{0}$$

Use the first identity proven in part (a) on the left side.

$$(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}$$

Suppose now that

$$(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}$$

Use the first identity proven in part (a) on the left side.

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u} = \mathbf{0}$$

The dot product is commutative.

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = \mathbf{0}$$

Multiply both sides by -1.

$$(\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = \mathbf{0}$$

Bring the first term to the right side.

$$-(\mathbf{v} \cdot \mathbf{w})\mathbf{u} = -(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Add $(\mathbf{u} \cdot \mathbf{w})\mathbf{v}$ to both sides.

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Use both results from part (a).

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

Therefore, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ if and only if $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}$.

Part (c)

Use the first result from part (a) to simplify the expression.

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = [(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}] + [(\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v}] + [(\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}]$$

Use the fact that the dot product is commutative in the first three terms on the right side.

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u} + (\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

All the terms cancel as a result.

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}$$