

Exercise 24

(a) Prove, without recourse to geometry, that

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \\ &= -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}).\end{aligned}$$

(b) Use part (a) and Exercise 23(a) to prove that

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u}' \times \mathbf{v}') &= (\mathbf{u} \cdot \mathbf{u}')(\mathbf{v} \cdot \mathbf{v}') - (\mathbf{u} \cdot \mathbf{v}')(\mathbf{u}' \cdot \mathbf{v}) \\ &= \begin{vmatrix} \mathbf{u} \cdot \mathbf{u}' & \mathbf{u} \cdot \mathbf{v}' \\ \mathbf{u}' \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v}' \end{vmatrix}.\end{aligned}$$

Solution

Part (a)

Let $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$ and $\mathbf{w} = (w_x, w_y, w_z)$.

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= (u_x, u_y, u_z) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \\ &= (u_x, u_y, u_z) \cdot [(v_y w_z - v_z w_y)\hat{\mathbf{x}} - (v_x w_z - v_z w_x)\hat{\mathbf{y}} + (v_x w_y - v_y w_x)\hat{\mathbf{z}}] \\ &= u_x(v_y w_z - v_z w_y) - u_y(v_x w_z - v_z w_x) + u_z(v_x w_y - v_y w_x) \\ &= u_x v_y w_z - u_x v_z w_y - u_y v_x w_z + u_y v_z w_x + u_z v_x w_y - u_z v_y w_x\end{aligned}$$

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) &= (v_x, v_y, v_z) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ w_x & w_y & w_z \\ u_x & u_y & u_z \end{vmatrix} \\ &= (v_x, v_y, v_z) \cdot [(w_y u_z - w_z u_y)\hat{\mathbf{x}} - (w_x u_z - w_z u_x)\hat{\mathbf{y}} + (w_x u_y - w_y u_x)\hat{\mathbf{z}}] \\ &= v_x(w_y u_z - w_z u_y) - v_y(w_x u_z - w_z u_x) + v_z(w_x u_y - w_y u_x) \\ &= u_z v_x w_y - u_y v_x w_z - u_z v_y w_x + u_x v_y w_z + u_y v_z w_x - u_x v_z w_y \\ &= u_x v_y w_z - u_x v_z w_y - u_y v_x w_z + u_y v_z w_x + u_z v_x w_y - u_z v_y w_x\end{aligned}$$

$$\begin{aligned}\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= (w_x, w_y, w_z) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \\ &= (w_x, w_y, w_z) \cdot [(u_y v_z - u_z v_y)\hat{\mathbf{x}} - (u_x v_z - u_z v_x)\hat{\mathbf{y}} + (u_x v_y - u_y v_x)\hat{\mathbf{z}}] \\ &= w_x(u_y v_z - u_z v_y) - w_y(u_x v_z - u_z v_x) + w_z(u_x v_y - u_y v_x) \\ &= u_y v_z w_x - u_z v_y w_x - u_x v_z w_y + u_z v_x w_y + u_x v_y w_z - u_y v_x w_z \\ &= u_x v_y w_z - u_x v_z w_y - u_y v_x w_z + u_y v_z w_x + u_z v_x w_y - u_z v_y w_x\end{aligned}$$

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) &= (u_x, u_y, u_z) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ w_x & w_y & w_z \\ v_x & v_y & v_z \end{vmatrix} \\
&= (u_x, u_y, u_z) \cdot [(w_y v_z - w_z v_y)\hat{\mathbf{x}} - (w_x v_z - w_z v_x)\hat{\mathbf{y}} + (w_x v_y - w_y v_x)\hat{\mathbf{z}}] \\
&= u_x(w_y v_z - w_z v_y) - u_y(w_x v_z - w_z v_x) + u_z(w_x v_y - w_y v_x) \\
&= u_x v_z w_y - u_x v_y w_z - u_y v_z w_x + u_y v_x w_z + u_z v_y w_x - u_z v_x w_y \\
&= -(u_x v_y w_z - u_x v_z w_y - u_y v_x w_z + u_y v_z w_x + u_z v_x w_y - u_z v_y w_x)
\end{aligned}$$

$$\begin{aligned}
\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) &= (w_x, w_y, w_z) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} \\
&= (w_x, w_y, w_z) \cdot [(v_y u_z - v_z u_y)\hat{\mathbf{x}} - (v_x u_z - v_z u_x)\hat{\mathbf{y}} + (v_x u_y - v_y u_x)\hat{\mathbf{z}}] \\
&= w_x(v_y u_z - v_z u_y) - w_y(v_x u_z - v_z u_x) + w_z(v_x u_y - v_y u_x) \\
&= u_z v_y w_x - u_y v_z w_x - u_z v_x w_y + u_x v_z w_y + u_y v_x w_z - u_x v_y w_z \\
&= -(u_x v_y w_z - u_x v_z w_y - u_y v_x w_z + u_y v_z w_x + u_z v_x w_y - u_z v_y w_x)
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) &= (v_x, v_y, v_z) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix} \\
&= (v_x, v_y, v_z) \cdot [(u_y w_z - u_z w_y)\hat{\mathbf{x}} - (u_x w_z - u_z w_x)\hat{\mathbf{y}} + (u_x w_y - u_y w_x)\hat{\mathbf{z}}] \\
&= v_x(u_y w_z - u_z w_y) - v_y(u_x w_z - u_z w_x) + v_z(u_x w_y - u_y w_x) \\
&= u_y v_x w_z - u_z v_x w_y - u_x v_y w_z + u_z v_y w_x + u_x v_z w_y - u_y v_z w_x \\
&= -(u_x v_y w_z - u_x v_z w_y - u_y v_x w_z + u_y v_z w_x + u_z v_x w_y - u_z v_y w_x)
\end{aligned}$$

Therefore, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$.

Part (b)

Treat $(\mathbf{u}' \times \mathbf{v}')$ as \mathbf{w} , use the fact that the dot product is commutative, use the result from part (a), and then use the second result from part (a) of Exercise 23.

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u}' \times \mathbf{v}') &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \\
&= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \\
&= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \\
&= \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{u}' \times \mathbf{v}')] \\
&= \mathbf{u} \cdot [(\mathbf{v} \cdot \mathbf{v}')\mathbf{u}' - (\mathbf{v} \cdot \mathbf{u}')\mathbf{v}'] \\
&= \mathbf{u} \cdot [\mathbf{u}'(\mathbf{v} \cdot \mathbf{v}') - \mathbf{v}'(\mathbf{u}' \cdot \mathbf{v})] \\
&= (\mathbf{u} \cdot \mathbf{u}')(\mathbf{v} \cdot \mathbf{v}') - (\mathbf{u} \cdot \mathbf{v}')(\mathbf{u}' \cdot \mathbf{v}) \\
&= \begin{vmatrix} \mathbf{u} \cdot \mathbf{u}' & \mathbf{u} \cdot \mathbf{v}' \\ \mathbf{u}' \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v}' \end{vmatrix}
\end{aligned}$$