

Exercise 7

What is the volume of the parallelepiped with sides $2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $5\mathbf{i} - 3\mathbf{k}$, and $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$?

Solution

Label each of the sides as

$$\mathbf{a} = (2, 1, -1)$$

$$\mathbf{b} = (5, 0, -3)$$

$$\mathbf{c} = (1, -2, 1)$$

The volume of the parallelepiped formed by these vectors is given by the triple product,

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (2, 1, -1) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 5 & 0 & -3 \\ 1 & -2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & -1 \\ 5 & 0 & -3 \\ 1 & -2 & 1 \end{vmatrix} \\ &= -5 \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + 0 - (-3) \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \\ &= -5[(1)(1) - (-1)(-2)] + 3[(2)(-2) - (1)(1)] \\ &= -10,\end{aligned}$$

or rather its magnitude: $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 10$.