Exercise 15

Show that in spherical coordinates:

(a) ρ is the length of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(b) $\phi = \cos^{-1}(\mathbf{v} \cdot \mathbf{k}/\|\mathbf{v}\|)$, where $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(c) $\theta = \cos^{-1}(\mathbf{u} \cdot \mathbf{i}/\|\mathbf{u}\|)$, where $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$.

Solution

The spherical coordinates of a point in space (x, y, z) are

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi,$$

where

$$\rho \geq 0, \qquad 0 \leq \theta < 2\pi, \qquad 0 \leq \phi \leq \pi.$$

Part (a)

Calculate the length, or magnitude, of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\begin{aligned}
\|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\| &= \sqrt{x^2 + y^2 + z^2} \\
&= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2} \\
&= \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi} \\
&= \sqrt{\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi} \\
&= \sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi} \\
&= \sqrt{\rho^2(\sin^2\phi + \cos^2\phi)} \\
&= \sqrt{\rho^2} \\
&= \rho
\end{aligned}$$

Part (b)

Define $\mathbf{v}$ as

$$\begin{aligned}
\mathbf{v} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\
&= \rho\sin\phi\cos\theta\,\mathbf{i} + \rho\sin\phi\sin\theta\,\mathbf{j} + \rho\cos\phi\,\mathbf{k}.
\end{aligned}$$

Take the dot product of both sides with $\mathbf{k}$.

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{k} &= (\rho\sin\phi\cos\theta\,\mathbf{i} + \rho\sin\phi\sin\theta\,\mathbf{j} + \rho\cos\phi\,\mathbf{k})\cdot\mathbf{k} \\
&= \rho\sin\phi\cos\theta(\mathbf{i}\cdot\mathbf{k}) + \rho\sin\phi\sin\theta(\mathbf{j}\cdot\mathbf{k}) + \rho\cos\phi(\mathbf{k}\cdot\mathbf{k}) \\
&= \rho\sin\phi\cos\theta(0) + \rho\sin\phi\sin\theta(0) + \rho\cos\phi(1) \\
&= \rho\cos\phi
\end{aligned}$$

In part (a) it was found that $\|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\| = \|\mathbf{v}\| = \rho$.

$$\mathbf{v} \cdot \mathbf{k} = \|\mathbf{v}\| \cos\phi$$

Divide both sides by $\|\mathbf{v}\|$.

$$\cos\phi = \frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\|}$$

Therefore,

$$\phi = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\|}\right).$$

Part (c)

Define $\mathbf{u}$ as

$$\begin{aligned}\mathbf{u} &= x\mathbf{i} + y\mathbf{j} \\ &= \rho\sin\phi\cos\theta\,\mathbf{i} + \rho\sin\phi\sin\theta\,\mathbf{j}.\end{aligned}$$

Take the dot product of both sides with $\mathbf{i}$.

$$\begin{aligned}\mathbf{u} \cdot \mathbf{i} &= (\rho\sin\phi\cos\theta\,\mathbf{i} + \rho\sin\phi\sin\theta\,\mathbf{j}) \cdot \mathbf{i} \\ &= \rho\sin\phi\cos\theta(\mathbf{i} \cdot \mathbf{i}) + \rho\sin\phi\sin\theta(\mathbf{j} \cdot \mathbf{i}) \\ &= \rho\sin\phi\cos\theta(1) + \rho\sin\phi\sin\theta(0) \\ &= \rho\sin\phi\cos\theta \qquad (1)\end{aligned}$$

Note that the length, or magnitude, of $\mathbf{u}$ is

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} \\ &= \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} \\ &= \sqrt{\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{\rho^2\sin^2\phi} \\ &= \rho\sin\phi.\end{aligned}$$

As a result, equation (1) becomes

$$\mathbf{u} \cdot \mathbf{i} = \|\mathbf{u}\| \cos\theta.$$

Divide both sides by $\|\mathbf{u}\|$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{i}}{\|\mathbf{u}\|}$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{i}}{\|\mathbf{u}\|}\right).$$