Exercise 3

(a) The following points are given in cylindrical coordinates; express each in rectangular coordinates and spherical coordinates: (1, 45°, 1), (2, $\pi/2$, −4), (0, 45°, 10), (3, $\pi/6$, 4), (1, $\pi/6$, 0), and (2, $3\pi/4$, −2). (Only the first point is solved in the Study Guide.)

(b) Change each of the following points from rectangular coordinates to spherical coordinates and to cylindrical coordinates: (2, 1, −2), (0, 3, 4), ($\sqrt{2}$, 1, 1), ($-2\sqrt{3}$, −2, 3). (Only the first point is solved in the Study Guide.)

Solution

Part (a)

Cartesian coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ), with ϕ being the polar angle, can be written in terms of cylindrical coordinates (r, θ, z) as

$$\begin{aligned} x &= r\cos\theta & \rho^2 &= r^2 + z^2 \\ y &= r\sin\theta & \theta &= \theta \\ z &= z & \rho\cos\phi &= z. \end{aligned}$$

$$(r = 1, \theta = 45°, z = 1)$$

$$\left.\begin{aligned} x &= 1\cos 45° \\ y &= 1\sin 45° \\ z &= 1 \end{aligned}\right\} \quad \rightarrow \quad \left(x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}, z = 1\right)$$

$$\left.\begin{aligned} &\rho = \sqrt{1^2 + 1^2} \\ &\theta = 45° \\ &\phi = \cos^{-1}\left(\frac{1}{\sqrt{1^2 + 1^2}}\right) \end{aligned}\right\} \quad \rightarrow \quad \left(\rho = \sqrt{2}, \theta = 45°, \phi = 45°\right)$$

$$(r = 2, \theta = \pi/2, z = -4)$$

$$\left.\begin{aligned} x &= 2\cos\frac{\pi}{2} \\ y &= 2\sin\frac{\pi}{2} \\ z &= -4 \end{aligned}\right\} \quad \rightarrow \quad (x = 0, y = 2, z = -4)$$

$$\left.\begin{aligned} &\rho = \sqrt{2^2 + (-4)^2} \\ &\theta = \frac{\pi}{2} \\ &\phi = \cos^{-1}\left(\frac{-4}{\sqrt{2^2 + (-4)^2}}\right) \end{aligned}\right\} \quad \rightarrow \quad \left(\rho = \sqrt{20}, \theta = \frac{\pi}{2}, \phi \approx 153°\right)$$

$$(r = 0, \theta = 45°, z = 10)$$

$$\left.\begin{aligned} x &= 0\cos 45° \\ y &= 0\sin 45° \\ z &= 10 \end{aligned}\right\} \quad \to \quad (x = 0, y = 0, z = 10)$$

$$\left.\begin{aligned} \rho &= \sqrt{0^2 + 10^2} \\ \theta &= 45° \\ \phi &= \cos^{-1}\left(\frac{10}{\sqrt{0^2 + 10^2}}\right) \end{aligned}\right\} \quad \to \quad (\rho = 10, \theta = 45°, \phi = 0)$$

$$(r = 3, \theta = \pi/6, z = 4)$$

$$\left.\begin{aligned} x &= 3\cos\frac{\pi}{6} \\ y &= 3\sin\frac{\pi}{6} \\ z &= 4 \end{aligned}\right\} \quad \to \quad \left(x = \frac{3\sqrt{3}}{2}, y = \frac{3}{2}, z = 4\right)$$

$$\left.\begin{aligned} \rho &= \sqrt{3^2 + 4^2} \\ \theta &= \frac{\pi}{6} \\ \phi &= \cos^{-1}\left(\frac{4}{\sqrt{3^2 + 4^2}}\right) \end{aligned}\right\} \quad \to \quad \left(\rho = 5, \theta = \frac{\pi}{6}, \phi \approx 36.9°\right)$$

$$(r = 1, \theta = \pi/6, z = 0)$$

$$\left.\begin{aligned} x &= 1\cos\frac{\pi}{6} \\ y &= 1\sin\frac{\pi}{6} \\ z &= 0 \end{aligned}\right\} \quad \to \quad \left(x = \frac{\sqrt{3}}{2}, y = \frac{1}{2}, z = 0\right)$$

$$\left.\begin{aligned} \rho &= \sqrt{1^2 + 0^2} \\ \theta &= \frac{\pi}{6} \\ \phi &= \cos^{-1}\left(\frac{0}{\sqrt{1^2 + 0^2}}\right) \end{aligned}\right\} \quad \to \quad \left(\rho = 1, \theta = \frac{\pi}{6}, \phi = \frac{\pi}{2}\right)$$

$$(r = 2, \theta = 3\pi/4, z = -2)$$

$$\left.\begin{aligned} x &= 2\cos\frac{3\pi}{4} \\ y &= 2\sin\frac{3\pi}{4} \\ z &= -2 \end{aligned}\right\} \quad \rightarrow \quad \left(x = -\sqrt{2}, y = \sqrt{2}, z = -2\right)$$

$$\left.\begin{aligned} \rho &= \sqrt{2^2 + (-2)^2} \\ \theta &= \frac{3\pi}{4} \\ \phi &= \cos^{-1}\left(\frac{-2}{\sqrt{2^2 + (-2)^2}}\right) \end{aligned}\right\} \quad \rightarrow \quad \left(\rho = \sqrt{8}, \theta = \frac{3\pi}{4}, \phi = \frac{3\pi}{4}\right)$$

Part (b)

Cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ), with ϕ being the polar angle, can be written in terms of Cartesian coordinates (x, y, z) as

$$\begin{aligned} r^2 &= x^2 + y^2 & \rho^2 &= x^2 + y^2 + z^2 \\ \tan\theta &= \frac{y}{x} & \tan\theta &= \frac{y}{x} \\ z &= z & \rho\cos\phi &= z. \end{aligned}$$

$$(x = 2, y = 1, z = -2)$$

$$\left.\begin{aligned} r &= \sqrt{2^2 + 1^2} \\ \theta &= \tan^{-1}\left(\frac{1}{2}\right) \\ z &= -2 \end{aligned}\right\} \quad \rightarrow \quad \left(r = \sqrt{5}, \theta \approx 26.6^\circ, z = -2\right)$$

$$\left.\begin{aligned} \rho &= \sqrt{2^2 + 1^2 + (-2)^2} \\ \theta &= \tan^{-1}\left(\frac{1}{2}\right) \\ \phi &= \cos^{-1}\left(\frac{-2}{\sqrt{2^2 + 1^2 + (-2)^2}}\right) \end{aligned}\right\} \quad \rightarrow \quad (\rho = 3, \theta \approx 26.6^\circ, \phi \approx 132^\circ)$$

$$(x = 0, y = 3, z = 4)$$

$$\left.\begin{aligned} r &= \sqrt{0^2 + 3^2} \\ \theta &= \tan^{-1}\left(\frac{3}{0}\right) \\ z &= 4 \end{aligned}\right\} \quad \rightarrow \quad \left(r = 3, \theta = \frac{\pi}{2}, z = 4\right)$$

$$\left.\begin{aligned} \rho &= \sqrt{0^2 + 3^2 + 4^2} \\ \theta &= \tan^{-1}\left(\frac{3}{0}\right) \\ \phi &= \cos^{-1}\left(\frac{4}{\sqrt{0^2 + 3^2 + 4^2}}\right) \end{aligned}\right\} \quad \rightarrow \quad \left(\rho = 5, \theta = \frac{\pi}{2}, \phi \approx 36.9^\circ\right)$$

$$(x = \sqrt{2}, y = 1, z = 1)$$

$$\left.\begin{aligned} r &= \sqrt{(\sqrt{2})^2 + 1^2} \\ \theta &= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ z &= 1 \end{aligned}\right\} \quad \rightarrow \quad \left(r = \sqrt{3}, \theta \approx 35.3^\circ, z = 1\right)$$

$$\left.\begin{aligned} \rho &= \sqrt{(\sqrt{2})^2 + 1^2 + 1^2} \\ \theta &= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ \phi &= \cos^{-1}\left(\frac{1}{\sqrt{(\sqrt{2})^2 + 1^2 + 1^2}}\right) \end{aligned}\right\} \quad \rightarrow \quad \left(\rho = 2, \theta \approx 35.3^\circ, \phi = \frac{\pi}{3}\right)$$

$$(x = -2\sqrt{3}, y = -2, z = 3)$$

$$\left.\begin{aligned} r &= \sqrt{(-2\sqrt{3})^2 + (-2)^2} \\ \theta &= \pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ z &= 3 \end{aligned}\right\} \quad \rightarrow \quad \left(r = 4, \theta = \frac{7\pi}{6}, z = 3\right)$$

$$\left.\begin{aligned} \rho &= \sqrt{(-2\sqrt{3})^2 + (-2)^2 + 3^2} \\ \theta &= \pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ \phi &= \cos^{-1}\left(\frac{3}{\sqrt{(-2\sqrt{3})^2 + (-2)^2 + 3^2}}\right) \end{aligned}\right\} \quad \rightarrow \quad \left(\rho = 5, \theta = \frac{7\pi}{6}, \phi \approx 53.1^\circ\right)$$