Exercise 1.1.6

Newton asked the question, what is the curve which, when revolved about an axis, gives a surface offering the least resistance to motion through a "rare" fluid (e.g. air)? Newton's answer is the following curve (see Chapter 7).

$$(x(t), y(t)) = \left(\frac{\lambda}{2}\left[\frac{1}{t} + 2t + t^3\right], \frac{\lambda}{2}\left[\ln\left(\frac{1}{t}\right) + t^2 + \frac{3}{4}t^4\right] - \frac{7}{8}\lambda\right),$$

where λ is a positive constant and $x \geq 2\lambda$. Graph this curve and compute its velocity and acceleration vectors. What angle does the curve make with the x-axis at the intersection point $(2\lambda, 0)$?

Solution

Setting $\lambda = 1$, the graph of this curve in the xy-plane is shown below.

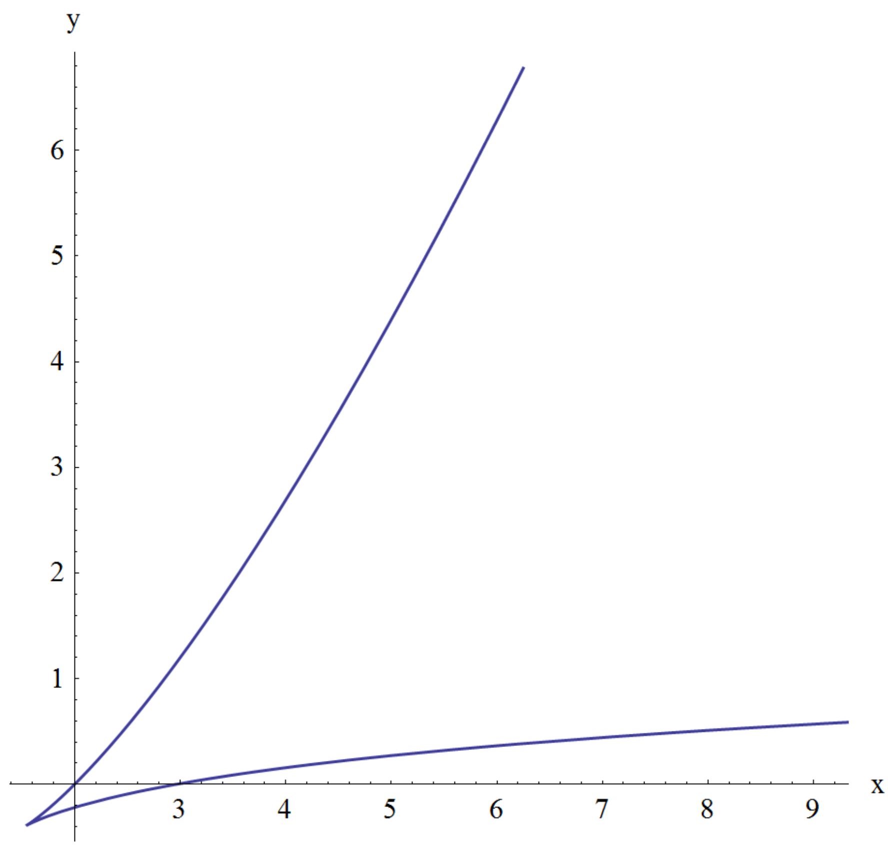

Figure 1: Plot of $\alpha(t) = (x(t), y(t))$ for $t \in (0, 2]$.

If this curve is revolved about the y-axis, the resulting surface is shown below.

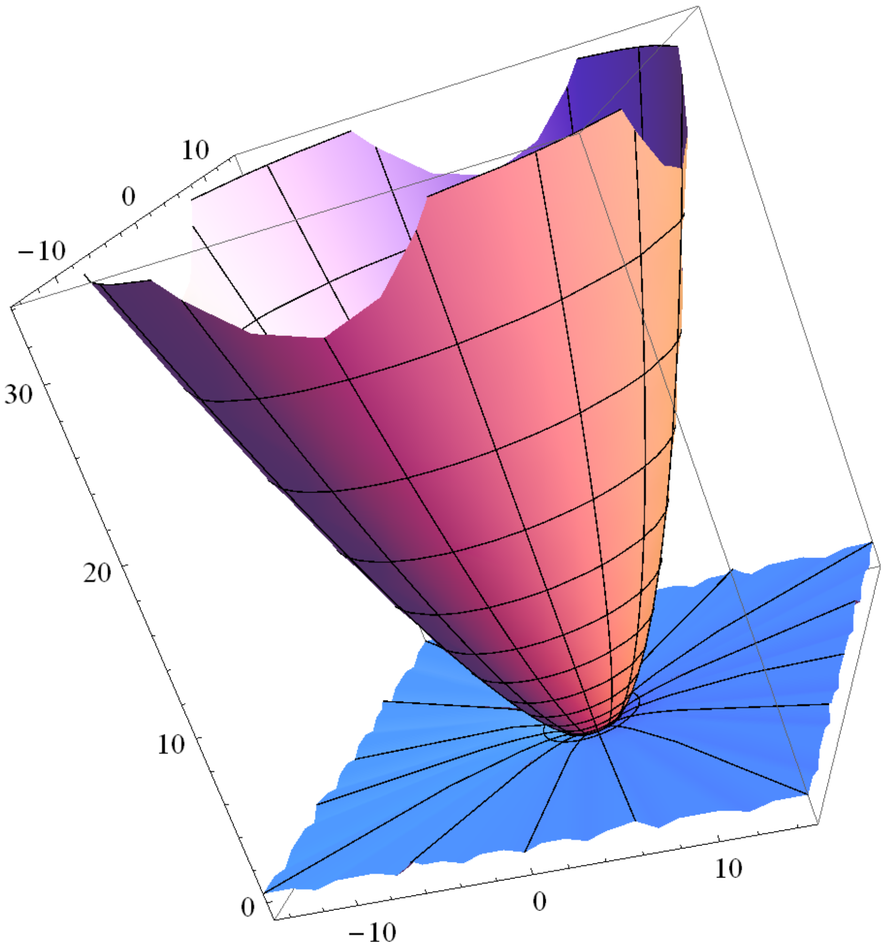

Figure 2: 3D Revolution Plot of $\alpha(t) = (x(t), y(t))$ for $t \in (0, 3]$.

Since $\alpha(t) = (x(t), y(t))$ is a curve in $\mathbb{R}^2$, its velocity and acceleration vectors are given by $\alpha'(t)$ and $\alpha''(t)$, respectively.

$$\alpha'(t) = \left(\frac{d}{dt}x(t), \frac{d}{dt}y(t)\right)$$

$$\alpha''(t) = \left(\frac{d^2}{dt^2}x(t), \frac{d^2}{dt^2}y(t)\right)$$

Differentiating both components of $\alpha(t)$ with respect to t, we get

$$\alpha'(t) = \left(\frac{\lambda}{2}\left(-\frac{1}{t^2} + 2 + 3t^2\right), \frac{\lambda}{2}\left(-\frac{1}{t} + 2t + 3t^3\right)\right).$$

Differentiating both components of $\alpha'(t)$ with respect to t, we get

$$\alpha''(t) = \left(\lambda\left(\frac{1}{t^3} + 3t\right), \frac{\lambda}{2}\left(\frac{1}{t^2} + 2 + 9t^2\right)\right).$$

In order to find the angle the curve makes with the x-axis, we have to find dy/dx at the given point. The slope is equal to $\tan\theta$, and θ is exactly what we need to solve for.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{\lambda}{2}\left(-\frac{1}{t^2} + 2 + 3t^2\right)}{\frac{\lambda}{2}\left(-\frac{1}{t} + 2t + 3t^3\right)} = \frac{1}{t}$$

Now we need to figure out what value of t gives $\alpha(t) = (2\lambda, 0)$. By inspection we see that $t = 1$ gives this result. Plugging this value of t into the formula for dy/dx gives

$$\left.\frac{dy}{dx}\right|_{t=1} = 1 = \tan\theta.$$

The value of θ that gives $\tan\theta = 1$ is $\theta = \pi/4$. Therefore, the angle the curve makes with the x-axis at $(2\lambda, 0)$ is $\pi/4$ radians.