Problem 1

If $f(x) = \sin(x^3)$, find $f^{(15)}(0)$.

Solution

The formula for the Taylor series of a function $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In order to answer the question, we'll have to figure out what the coefficient is of the x^{15} term. The Taylor series of $\sin x$ centered at $x = 0$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

To get the Taylor series for $\sin x^3$ centered at $x = 0$, simply replace x with x^3 in the formula.

$$\sin x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^3)^{2n+1}$$

Hence,

$$\sin x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{6n+3}.$$

To get 15 in the exponent of x, we have to set n equal to 2. Plugging in $n = 2$ yields

$$\frac{(-1)^2}{5!}.$$

for the coefficient of x^{15}. According to the definition of the Taylor series, this has to be equal to

$$\frac{f^{(15)}(0)}{15!}.$$

We thus have an equation we can use to answer the question.

$$\frac{f^{(15)}(0)}{15!} = \frac{1}{5!}$$

Therefore,

$$f^{(15)}(0) = \frac{15!}{5!} = 10\,897\,286\,400.$$