Problem 10

If $a_0 + a_1 + a_2 + \cdots + a_k = 0$, show that

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = 0$$

If you don't see how to prove this, try the problem-solving strategy of *using analogy* (see page 71). Try the special cases $k = 1$ and $k = 2$ first. If you can see how to prove the assertion for these cases, then you will probably see how to prove it in general.

Solution

Factor out $\sqrt{n}$.

$$\lim_{n\to\infty}\sqrt{n}\left(a_0 + a_1\frac{\sqrt{n+1}}{\sqrt{n}} + a_2\frac{\sqrt{n+2}}{\sqrt{n}} + \cdots + a_k\frac{\sqrt{n+k}}{\sqrt{n}}\right)$$

Combine the square roots and split the fractions up.

$$\lim_{n\to\infty}\sqrt{n}\left(a_0 + a_1\sqrt{1+\frac{1}{n}} + a_2\sqrt{1+\frac{2}{n}} + \cdots + a_k\sqrt{1+\frac{k}{n}}\right)$$

Because $a_0 + a_1 + a_2 + \cdots + a_k = 0$, what we have here is $\infty \cdot 0$ as $n \to \infty$. Change this to the $0/0$ indeterminate form.

$$\lim_{n\to\infty}\frac{a_0 + a_1\sqrt{1+\frac{1}{n}} + a_2\sqrt{1+\frac{2}{n}} + \cdots + a_k\sqrt{1+\frac{k}{n}}}{\frac{1}{\sqrt{n}}}$$

Now apply L'Hôpital's rule.

$$\lim_{n\to\infty}\frac{a_1\cdot\frac{1}{2}\left(1+\frac{1}{n}\right)^{-1/2}\left(-\frac{1}{n^2}\right) + a_2\cdot\frac{1}{2}\left(1+\frac{2}{n}\right)^{-1/2}\left(-\frac{2}{n^2}\right) + \cdots + a_k\cdot\frac{1}{2}\left(1+\frac{k}{n}\right)^{-1/2}\left(-\frac{k}{n^2}\right)}{-\frac{1}{2}n^{-3/2}}$$

All the minus signs and 1/2's cancel out.

$$\lim_{n\to\infty} n^{3/2}\left[a_1\left(1+\frac{1}{n}\right)^{-1/2}\left(\frac{1}{n^2}\right) + a_2\left(1+\frac{2}{n}\right)^{-1/2}\left(\frac{2}{n^2}\right) + \cdots + a_k\left(1+\frac{k}{n}\right)^{-1/2}\left(\frac{k}{n^2}\right)\right]$$

Distribute the $n^{3/2}$ to every term.

$$\lim_{n\to\infty}\left(\frac{a_1}{\sqrt{1+\frac{1}{n}}\sqrt{n}} + \frac{a_2}{\sqrt{1+\frac{2}{n}}\sqrt{n}} + \cdots + \frac{a_k}{\sqrt{1+\frac{k}{n}}\sqrt{n}}\right)$$

Combine the square roots.

$$\lim_{n\to\infty}\left(\frac{a_1}{\sqrt{n+1}}+\frac{a_2}{\sqrt{n+2}}+\cdots+\frac{a_k}{\sqrt{n+k}}\right)$$

As n goes to infinity, every term goes to 0. Therefore,

$$\lim_{n\to\infty}\left(a_0\sqrt{n}+a_1\sqrt{n+1}+a_2\sqrt{n+2}+\cdots+a_k\sqrt{n+k}\right)=0.$$