Problem 14

If $p > 1$, evaluate the expression

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

Solution

Since $1^p = 1$, the first term in the numerator and denominator can be written as a fraction depending on p like the others.

$$\frac{\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

The series in the numerator is straightforward to write compactly.

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

The series in the denominator is clearly alternating; however, we will not write it with a $(-1)^n$ term. Rather, we will write it like so.

$$\begin{aligned}
\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots &= \left(\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right) - 2\left(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \cdots\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^p} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^p} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^p} - \frac{2}{2^p}\sum_{n=1}^{\infty} \frac{1}{n^p} \\
&= \left(1 - \frac{2}{2^p}\right)\sum_{n=1}^{\infty} \frac{1}{n^p}
\end{aligned}$$

The point of writing it like this is so that now the series will cancel each other out in the fraction.

$$\frac{\cancel{\sum_{n=1}^{\infty} \frac{1}{n^p}}}{\left(1 - \frac{2}{2^p}\right)\cancel{\sum_{n=1}^{\infty} \frac{1}{n^p}}} = \frac{1}{1 - \frac{2}{2^p}}$$

Multiply the numerator and denominator by 2^p to eliminate the fraction on the bottom.

$$\frac{1}{1 - \frac{2}{2^p}} \cdot \frac{2^p}{2^p} = \frac{2^p}{2^p - 2}$$

Therefore,

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \frac{2^p}{2^p - 2}, \quad p > 1.$$