Problem 19

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)3^n}$.

Solution

Looking at the table of Taylor series on page 768, we see that this series looks very similar to the one for $\tan^{-1} x$.

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

The objective here is to write the given series in this form. Start off by beginning the sum from $n = 0$. Plugging in $n = 0$ to the summand gives us 1, so subtract it off.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)3^n} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} - 1 \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{3}\right)^n}{2n+1} - 1 \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{\sqrt{3}}\right)^{2n}}{2n+1} - 1 \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}\cdot\sqrt{3}}{2n+1} - 1 \\
&= \sqrt{3}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1} - 1 \\
&= \sqrt{3}\cdot\tan^{-1}\frac{1}{\sqrt{3}} - 1 \\
&= \sqrt{3}\cdot\frac{\pi}{6} - 1
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = \frac{\pi\sqrt{3}-6}{6}.$$