Problem 22

Right-angled triangles are constructed as in the figure. Each triangle has height 1 and its base is the hypotenuse of the preceding triangle. Show that this sequence of triangles makes indefinitely many turns around P by showing that $\sum \theta_n$ is a divergent series.

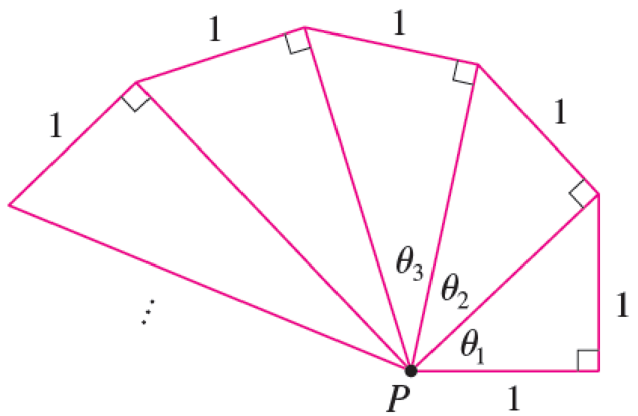

FIGURE FOR PROBLEM 22

Solution

The first objective is to find an expression for θ_n. We do this by considering the nth triangle.

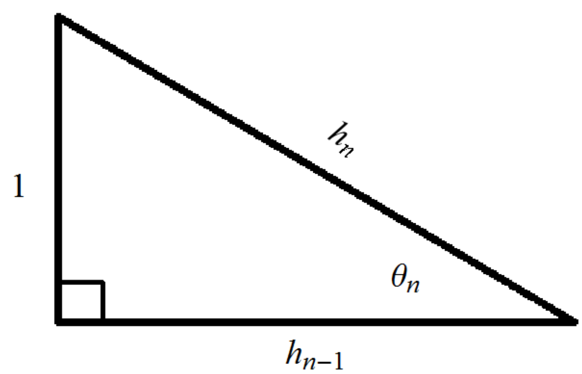

Figure 1: The nth triangle.

This is a right triangle, so we can use trigonometry to relate θ_n to the sides. We'll use tangent since the only Taylor series for inverse functions available on page 768 is $\tan^{-1} x$.

$$\tan \theta_n = \frac{1}{h_{n-1}}$$

We determine h_{n-1} by finding h_1, then h_2, and trying to figure out a pattern.

$$\begin{aligned}
h_1^2 &= 1^2 + 1^2 = 2 & &\to & h_1 &= \sqrt{2} \\
h_2^2 &= 1^2 + h_1^2 = 1^2 + 2 = 3 & &\to & h_2 &= \sqrt{3} \\
h_3^2 &= 1^2 + h_2^2 = 1^2 + 3 = 4 & &\to & h_3 &= \sqrt{4} \\
h_4^2 &= 1^2 + h_3^2 = 1^2 + 4 = 5 & &\to & h_4 &= \sqrt{5}
\end{aligned}$$

We can see that $h_{n-1} = \sqrt{n}$. Thus,

$$\tan\theta_n = \frac{1}{\sqrt{n}}$$

and

$$\theta_n = \tan^{-1}\frac{1}{\sqrt{n}}.$$

We now have to show that

$$\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{\sqrt{n}}$$

is a divergent series. The Taylor series for $\tan^{-1} x$ is given on page 768 as

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Plug in $1/\sqrt{n}$ for x.

$$\begin{aligned}
\tan^{-1}\frac{1}{\sqrt{n}} &= \frac{1}{\sqrt{n}} - \frac{1}{3}\left(\frac{1}{\sqrt{n}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{n}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{n}}\right)^7 + \cdots \\
&= \frac{1}{n^{1/2}} - \frac{1}{3}\cdot\frac{1}{n^{3/2}} + \frac{1}{5}\cdot\frac{1}{n^{5/2}} - \frac{1}{7}\cdot\frac{1}{n^{7/2}} + \cdots
\end{aligned}$$

Substituting the Taylor expansion into the series gives us

$$\begin{aligned}
\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{\sqrt{n}} &= \sum_{n=1}^{\infty}\left(\frac{1}{n^{1/2}} - \frac{1}{3}\cdot\frac{1}{n^{3/2}} + \frac{1}{5}\cdot\frac{1}{n^{5/2}} - \frac{1}{7}\cdot\frac{1}{n^{7/2}} + \cdots\right) \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1/2}} - \sum_{n=1}^{\infty}\frac{1}{3}\cdot\frac{1}{n^{3/2}} + \sum_{n=1}^{\infty}\frac{1}{5}\cdot\frac{1}{n^{5/2}} - \sum_{n=1}^{\infty}\frac{1}{7}\cdot\frac{1}{n^{7/2}} + \cdots \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1/2}} - \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} + \frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{n^{5/2}} - \frac{1}{7}\sum_{n=1}^{\infty}\frac{1}{n^{7/2}} + \cdots
\end{aligned}$$

All of these series on the right side are p-series, which have the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

and only converge if $p > 1$. The first series has $p = 1/2$, which means it is a divergent series. Therefore,

$$\sum_{n=1}^{\infty}\theta_n$$

is a divergent series, which means the sequence of triangles makes indefinitely many turns around P.