Problem 6

Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

Solution

Rewrite the series in terms of the prime factors, 2 and 3.

$$\frac{1}{2^0 3^0} + \frac{1}{2^1 3^0} + \frac{1}{2^0 3^1} + \frac{1}{2^1 3^1} + \frac{1}{2^0 3^2} + \frac{1}{2^2 3^0} + \frac{1}{2^2 3^1} + \frac{1}{2^1 3^2} + \frac{1}{2^2 3^2} + \cdots$$

The aim here is to write this series compactly. This can be done conveniently using two double series.

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{1}{2^p 3^{p-k}} = \underbrace{\frac{1}{2^0 3^0}}_{p=0} + \underbrace{\frac{1}{2^1 3^1} + \frac{1}{2^1 3^0}}_{p=1} + \underbrace{\frac{1}{2^2 3^2} + \frac{1}{2^2 3^1} + \frac{1}{2^2 3^0}}_{p=2} + \underbrace{\frac{1}{2^3 3^3} + \frac{1}{2^3 3^2} + \frac{1}{2^3 3^1} + \frac{1}{2^3 3^0}}_{p=3} + \cdots$$

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{1}{2^{p-k} 3^{p}} = \underbrace{\frac{1}{2^0 3^0}}_{p=0} + \underbrace{\frac{1}{2^1 3^1} + \frac{1}{2^0 3^1}}_{p=1} + \underbrace{\frac{1}{2^2 3^2} + \frac{1}{2^1 3^2} + \frac{1}{2^0 3^2}}_{p=2} + \underbrace{\frac{1}{2^3 3^3} + \frac{1}{2^2 3^3} + \frac{1}{2^1 3^3} + \frac{1}{2^3 3^0}}_{p=3} + \cdots$$

In a double series a particular value for p is chosen and then k gets summed through those values. In the first double series, for example, p starts at 0 and k takes one value, $k = 0$. Then p goes to 1 and k takes two values, $k = 0$ and $k = 1$. Then p goes to 2 and k takes three values, $k = 0$ and $k = 1$ and $k = 2$. The reason we need the second double series is because the first one doesn't cover the fractions where the exponent of 3 is higher than that of 2. Notice that the terms where the exponents are the same appear in both double series. If we add the two double series together to obtain the original series, then we have to subtract these repeated terms so we don't count them twice.

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{1}{2^p 3^{p-k}} + \sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{1}{2^{p-k} 3^{p}} - \sum_{p=0}^{\infty} \frac{1}{2^p 3^p}$$

Now that the original series has been written compactly, we can evaluate it. Start by combining the double series and writing the single sum like so.

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p} \left(\frac{1}{2^p 3^{p-k}} + \frac{1}{2^{p-k} 3^{p}} \right) - \sum_{p=0}^{\infty} \left(\frac{1}{6} \right)^p$$

The single sum is an infinite geometric series, which can be evaluated. Recall that the formula for it is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r},$$

where it converges if $|r| < 1$. So we have the following.

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p} \left(\frac{3^k}{2^p 3^p} + \frac{2^k}{2^p 3^p} \right) - \frac{1}{1-\frac{1}{6}}$$

Simplify the terms.

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p}\left(\frac{3^k}{6^p}+\frac{2^k}{6^p}\right)-\frac{6}{5}$$

The index of the inner sum is k, so any terms with p are constant and can be pulled out in front of it.

$$\sum_{p=0}^{\infty}\frac{1}{6^p}\sum_{k=0}^{p}\left(3^k+2^k\right)-\frac{6}{5}$$

Distribute the inner sum to both terms.

$$\sum_{p=0}^{\infty}\frac{1}{6^p}\left(\sum_{k=0}^{p}3^k+\sum_{k=0}^{p}2^k\right)-\frac{6}{5}$$

In the parentheses we have finite geometric series. Recall that the formula for one is

$$\sum_{k=0}^{p}a_1r^k=a_1\cdot\frac{1-r^{p+1}}{1-r},$$

where $r\neq 0,1$. So we have the following.

$$\sum_{p=0}^{\infty}\frac{1}{6^p}\left(\frac{3^{p+1}-1}{2}+\frac{2^{p+1}-1}{1}\right)-\frac{6}{5}$$

Distribute the $1/6^p$ and write the terms like so.

$$\sum_{p=0}^{\infty}\left(\frac{3}{2}\cdot\frac{3^p}{6^p}-\frac{1}{2}\cdot\frac{1}{6^p}+\frac{2}{1}\cdot\frac{2^p}{6^p}-\frac{1}{6^p}\right)-\frac{6}{5}$$

Distribute the sum to every term.

$$\sum_{p=0}^{\infty}\frac{3}{2}\cdot\frac{1}{2^p}-\sum_{p=0}^{\infty}\frac{1}{2}\cdot\frac{1}{6^p}+\sum_{p=0}^{\infty}2\cdot\frac{1}{3^p}-\sum_{p=0}^{\infty}\frac{1}{6^p}-\frac{6}{5}$$

Bring the constants out in front.

$$\frac{3}{2}\sum_{p=0}^{\infty}\left(\frac{1}{2}\right)^p-\frac{1}{2}\sum_{p=0}^{\infty}\left(\frac{1}{6}\right)^p+2\sum_{p=0}^{\infty}\left(\frac{1}{3}\right)^p-\sum_{p=0}^{\infty}\left(\frac{1}{6}\right)^p-\frac{6}{5}$$

All of these infinite series are geometric and can be evaluated.

$$\frac{3}{2}\cdot\frac{1}{1-\frac{1}{2}}-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{6}}+2\cdot\frac{1}{1-\frac{1}{3}}-\frac{1}{1-\frac{1}{6}}-\frac{6}{5}$$

$$3-\frac{3}{5}+3-\frac{6}{5}-\frac{6}{5}=6-3=3$$

Therefore,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\cdots=3.$$