Problem 7

(a) Show that for $xy \neq 1$,

$$\arctan x - \arctan y = \arctan \frac{x - y}{1 + xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

(b) Show that $\arctan \frac{120}{119} - \arctan \frac{1}{239} = \pi/4$.

(c) Deduce the following formula of John Machin (1680-1751):

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

(d) Use the Maclaurin series for arctan to show that

$$0.1973955597 < \arctan \frac{1}{5} < 0.1973955616$$

(e) Show that

$$0.004184075 < \arctan \frac{1}{239} < 0.004184077$$

(f) Deduce that, correct to seven decimal places, $\pi \approx 3.1415927$.

Machin used this method in 1706 to find π correct to 100 decimal places. Recently, with the aid of computers, the value of π has been computed to increasingly greater accuracy. In 2013 Shigeru Kondo and Alexander Yee computed the value of π to more than 12 trillion decimal places!

Solution

Part (a)

We'll make use of the angle subtraction formula for tangent, namely that

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$$

Hence, taking the tangent of the left-hand side gives us

$$\tan(\arctan x - \arctan y) = \frac{\tan \arctan x - \tan \arctan y}{1 + \tan \arctan x \tan \arctan y}$$

Since the tangent and arctangent are inverse functions, they cancel each other out.

$$\tan(\arctan x - \arctan y) = \frac{x - y}{1 + xy}$$

Take the arctangent of both sides to get the desired result.

$$\arctan x - \arctan y = \arctan \frac{x - y}{1 + xy}$$

Part (b)

Here we make use of the identity we just proved.

$$\begin{aligned}
\arctan\frac{120}{119} - \arctan\frac{1}{239} &= \arctan\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}} \\
&= \arctan\frac{\frac{120\cdot 239 - 119}{119\cdot 239}}{\frac{119\cdot 239 + 120}{119\cdot 239}} \\
&= \arctan\frac{28561}{28561} \\
&= \arctan 1
\end{aligned}$$

Therefore,

$$\arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4}.$$

Part (c)

The only difference between this formula and the one in part (b) is that we have $4\arctan(1/5)$ instead of $\arctan(120/119)$. Our aim then is to show that

$$\arctan\frac{120}{119} = 4\arctan\frac{1}{5}.$$

We'll use the double angle formula for tangent to do this, which says that

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Consider the tangent of the right-hand side.

$$\tan\left(4\arctan\frac{1}{5}\right)$$

Applying the double angle formula to this gives us the following.

$$\begin{aligned}
\tan\left(4\arctan\frac{1}{5}\right) &= \frac{2\tan\left(2\arctan\frac{1}{5}\right)}{1 - \tan^2\left(2\arctan\frac{1}{5}\right)} \\
&= \frac{2\tan\left(2\arctan\frac{1}{5}\right)}{1 - \left[\tan\left(2\arctan\frac{1}{5}\right)\right]\left[\tan\left(2\arctan\frac{1}{5}\right)\right]}
\end{aligned}$$

Apply the double angle formula three times now: once in the numerator and twice in the denominator.

$$\begin{aligned}
\tan\left(4\arctan\frac{1}{5}\right) &= \frac{2\cdot\frac{2\tan(\arctan\frac{1}{5})}{1-\tan^2(\arctan\frac{1}{5})}}{1 - \left[\frac{2\tan(\arctan\frac{1}{5})}{1-\tan^2(\arctan\frac{1}{5})}\right]\left[\frac{2\tan(\arctan\frac{1}{5})}{1-\tan^2(\arctan\frac{1}{5})}\right]} \\
&= \frac{4\cdot\frac{\frac{1}{5}}{1-(\frac{1}{5})^2}}{1 - 4\left[\frac{\frac{1}{5}}{1-(\frac{1}{5})^2}\right]\left[\frac{\frac{1}{5}}{1-(\frac{1}{5})^2}\right]} \\
&= \frac{\frac{5}{6}}{\frac{119}{144}} \\
&= \frac{120}{119}
\end{aligned}$$

Hence,

$$\tan\left(4\arctan\frac{1}{5}\right) = \frac{120}{119}.$$

Take the arctangent of both sides to get the desired result,

$$4\arctan\frac{1}{5} = \arctan\frac{120}{119}.$$

Therefore,

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \frac{\pi}{4}.$$

Part (d)

According to the table on page 768, the Maclaurin series for arctan is this.

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

We have to add the first six terms to get the specified lower bound and the first seven terms to get the specified upper bound.

$$\sum_{n=0}^{5} \frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} \approx 0.19739555975 > 0.1973955597$$

$$\sum_{n=0}^{6} \frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} \approx 0.197395559852 < 0.1973955616$$

So we have

$$0.1973955597 < \sum_{n=0}^{5} \frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} < \arctan\frac{1}{5} < \sum_{n=0}^{6} \frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} < 0.1973955616.$$

Therefore,

$$0.1973955597 < \arctan\frac{1}{5} < 0.1973955616.$$

Part (e)

Because of how small 1/239 is, we only need to add the first two terms to get the specified lower bound and the first three terms to get the specified upper bound.

$$\sum_{n=0}^{1} \frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1} \approx 0.0041840760018 > 0.004184075$$

$$\sum_{n=0}^{2} \frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1} \approx 0.0041840760020 < 0.004184077$$

So we have

$$0.004184075 < \sum_{n=0}^{1} \frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1} < \arctan\frac{1}{239} < \sum_{n=0}^{2} \frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1} < 0.004184077.$$

Therefore,

$$0.004184075 < \arctan\frac{1}{239} < 0.004184077.$$

Part (f)

We'll make use of Mr. Machin's formula to calculate π.

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \frac{\pi}{4}$$

Multiply both sides by 4.

$$\pi = 4\left(4\arctan\frac{1}{5} - \arctan\frac{1}{239}\right)$$

In order to get the first seven decimal places of π, we only need the first five terms of the first arctan function and the first two terms of the second one.

$$\begin{aligned}\pi &\approx 4\left[4\sum_{n=0}^{4}\frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} - \sum_{n=0}^{1}\frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1}\right] \\ &\approx 4\left[4\left(\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \frac{1}{7}\cdot\frac{1}{5^7} + \frac{1}{9}\cdot\frac{1}{5^9}\right) - \left(\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{239^3}\right)\right] \\ &\approx 3.141592682\end{aligned}$$

Therefore, correct to seven decimal places,

$$\pi \approx 3.1415927.$$