Problem 11

If $0 < a < b$, find

$$\lim_{t\to 0}\left\{\int_0^1 [bx + a(1-x)]^t\,dx\right\}^{1/t}$$

Solution

We will solve the integral first and then evaluate the limit. In the integrand there's a function (enclosed in the square brackets) inside of another function (power function). We can solve this with a u-substitution, letting u be equal to the inner function.

$$u = bx + a(1-x)$$

$$du = (b-a)\,dx \quad \to \quad \frac{du}{b-a} = dx$$

Making this substitution changes not only the integrand but also the limits. The old limits are $x = 0$ and $x = 1$. Plug these values into the u-substitution to obtain the new limits. Doing so gives $u = a$ and $u = b$, respectively.

$$\lim_{t\to 0}\left\{\int_a^b u^t\,\frac{du}{b-a}\right\}^{1/t}$$

$1/(b-a)$ is a constant, so it can be pulled out in front of the integral.

$$\lim_{t\to 0}\left\{\frac{1}{b-a}\int_a^b u^t\,du\right\}^{1/t}$$

Evaluate the integral. Because it's a power function, bump up the power by 1 and divide by that same number.

$$\lim_{t\to 0}\left\{\frac{1}{b-a}\cdot\left.\frac{u^{t+1}}{t+1}\right|_a^b\right\}^{1/t}$$

Plug in the limits of integration.

$$\lim_{t\to 0}\left\{\frac{1}{b-a}\cdot\frac{1}{t+1}(b^{t+1}-a^{t+1})\right\}^{1/t}$$

The integral is gone, and all that's left is to determine the limit. Since there's a $1/t$ in the exponent, we'll need to use the logarithm to bring it down. Make use of the following trick.

$$\lim_{t\to 0} e^{\ln\left\{\frac{1}{b-a}\cdot\frac{1}{t+1}(b^{t+1}-a^{t+1})\right\}^{1/t}}$$

Now that there's a logarithm in front, the exponent can become the coefficient.

$$\lim_{t\to 0} e^{\frac{1}{t}\ln\left\{\frac{1}{b-a}\cdot\frac{1}{t+1}(b^{t+1}-a^{t+1})\right\}}$$

Bring the limit into the exponent.

$$e^{\lim_{t\to 0} \frac{1}{t} \ln\left\{\frac{1}{b-a}\cdot\frac{1}{t+1}(b^{t+1}-a^{t+1})\right\}}$$

Plugging in $t=0$ gives $\ln 1$ (which is 0) over 0. This is an indeterminate form, so l'Hôpital's rule can be applied here. The derivative of t is just 1, so the denominator disappears. We essentially replace the entire expression with the derivative of the ln term. We have to use the chain rule here because there's a whole function of t inside the logarithm; that is, the function's derivative has to be multiplied as well.

$$e^{\lim_{t\to 0} \left\{\frac{b-a}{b^{t+1}-a^{t+1}}\cdot\frac{t+1}{1}\cdot\frac{d}{dt}\left[\frac{1}{b-a}\cdot\frac{1}{t+1}(b^{t+1}-a^{t+1})\right]\right\}}$$

$1/(b-a)$ comes out of the derivative and cancels with the $b-a$. Use the quotient rule to evaluate the derivative. Note that $d/dt(t^{a+1})=(a+1)t^a$, but $d/dt(a^{t+1})=a^{t+1}\ln a$.

$$e^{\lim_{t\to 0} \left\{\frac{t+1}{b^{t+1}-a^{t+1}}\cdot\frac{(b^{t+1}\ln b-a^{t+1}\ln a)(t+1)-1\cdot(b^{t+1}-a^{t+1})}{(t+1)^2}\right\}}$$

Simplify the resulting expression.

$$e^{\lim_{t\to 0} \left\{\frac{1}{b^{t+1}-a^{t+1}}\cdot\frac{(b^{t+1}\ln b-a^{t+1}\ln a)(t+1)-b^{t+1}+a^{t+1}}{t+1}\right\}}$$

Now we can plug in $t=0$ to evaluate the limit. Doing so gives

$$e^{\left[\frac{1}{b-a}\cdot\frac{(b\ln b-a\ln a)-b+a}{1}\right]}.$$

Distribute the $1/(b-a)$.

$$e^{\left[\frac{1}{b-a}(b\ln b-a\ln a)-1\right]}$$

Combine the logarithms into one.

$$e^{\left[\frac{1}{b-a}\ln\frac{b^b}{a^a}-1\right]}$$

Bring up the coefficient to the exponent.

$$e^{\left[\ln\left(\frac{b^b}{a^a}\right)^{1/(b-a)}-1\right]}$$

Split up the exponential into two.

$$e^{\ln\left(\frac{b^b}{a^a}\right)^{1/(b-a)}} e^{-1}$$

Therefore,

$$\lim_{t\to 0}\left\{\int_0^1 [bx+a(1-x)]^t\,dx\right\}^{1/t} = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}.$$