Problem 16

A rocket is fired straight up, burning fuel at the constant rate of b kilograms per second. Let $v = v(t)$ be the velocity of the rocket at time t and suppose that the velocity u of the exhaust gas is constant. Let $M = M(t)$ be the mass of the rocket at time t and note that M decreases as the fuel burns. If we neglect air resistance, it follows from Newton's Second Law that

$$F = M\frac{dv}{dt} - ub$$

where the force $F = -Mg$. Thus

$$M\frac{dv}{dt} - ub = -Mg \tag{1}$$

Let M_1 be the mass of the rocket without fuel, M_2 the initial mass of the fuel, and $M_0 = M_1 + M_2$. Then, until the fuel runs outs at time $t = M_2/b$, the mass is $M = M_0 - bt$.

(a) Substitute $M = M_0 - bt$ into Equation 1 and solve the resulting equation for v. Use the initial condition $v(0) = 0$ to evaluate the constant.

(b) Determine the velocity of the rocket at time $t = M_2/b$. This is called the *burnout velocity*.

(c) Determine the height of the rocket $y = y(t)$ at the burnout time.

(d) Find the height of the rocket at any time t.

Solution

Part (a)

Substitute $M = M_0 - bt$ into Equation 1.

$$(M_0 - bt)\frac{dv}{dt} - ub = -(M_0 - bt)g$$

We want to isolate the term that contains v, so bring ub to the right side.

$$(M_0 - bt)\frac{dv}{dt} = -(M_0 - bt)g + ub$$

Divide both sides by $M_0 - bt$.

$$\frac{dv}{dt} = \frac{-(M_0 - bt)g + ub}{M_0 - bt}$$

Multiply both sides by dt.

$$dv = \frac{-(M_0 - bt)g + ub}{M_0 - bt}\,dt$$

Integrate both sides.

$$\int dv = \int \frac{-(M_0 - bt)g + ub}{M_0 - bt}\,dt$$

Split up the fraction into two.

$$v = \int \left[\frac{-(M_0 - bt)g}{M_0 - bt} + \frac{ub}{M_0 - bt}\right] dt$$

Cancel $M_0 - bt$.

$$v = \int \left(-g + \frac{ub}{M_0 - bt}\right) dt$$

Split up the integral into two.

$$v = \int (-g)\, dt + \int \frac{ub}{M_0 - bt}\, dt$$

Use a substitution for the second integral.

$$w = M_0 - bt$$
$$dw = -b\, dt \quad \rightarrow \quad -dw = b\, dt$$

The first integral can be evaluated easily since g is constant.

$$v = -gt + \int \frac{u}{w} (-dw) + C$$

Bring the constants in front of the integral.

$$v = -gt - u \int \frac{1}{w}\, dw + C$$

Now the second integral can be determined.

$$v = -gt - u \ln|w| + D$$

Since we want v in terms of t, plug back in the expression for w.

$$v(t) = -gt - u \ln|M_0 - bt| + D$$

Since $M_0 - bt$ represents the mass of the rocket, it can never be negative, so the absolute value sign can be dropped.

$$v(t) = -gt - u \ln(M_0 - bt) + D$$

Use the initial condition $v(0) = 0$ to determine the constant of integration D.

$$v(0) = -u \ln M_0 + D = 0 \quad \rightarrow \quad D = u \ln M_0$$

Substitute this expression for D into the equation for $v(t)$.

$$v(t) = -gt - u \ln(M_0 - bt) + u \ln M_0$$

Factor u from the last two terms.

$$v(t) = -gt + u[\ln M_0 - \ln(M_0 - bt)]$$

Combine the logarithms. Therefore,

$$v(t) = -gt + u \ln \frac{M_0}{M_0 - bt}.$$

Part (b)

Here we have to evaluate $v(t)$ when $t = M_2/b$.

$$v\left(t = \frac{M_2}{b}\right) = -g\frac{M_2}{b} + u\ln\frac{M_0}{M_0 - b\cdot\frac{M_2}{b}}$$

Cancel b.

$$v\left(t = \frac{M_2}{b}\right) = -\frac{g}{b}M_2 + u\ln\frac{M_0}{M_0 - M_2}$$

Since $M_0 = M_1 + M_2$, we have $M_0 - M_2 = M_1$. Therefore,

$$v\left(t = \frac{M_2}{b}\right) = -\frac{g}{b}M_2 + u\ln\frac{M_0}{M_1}.$$

Part (c)

The formula for the height as derived in part (d) is

$$y(t) = -\frac{1}{2}gt^2 + \frac{u}{b}(M_0 - bt)\ln\frac{M_0 - bt}{M_0} + ut.$$

Substitute $t = M_2/b$.

$$y\left(t = \frac{M_2}{b}\right) = -\frac{1}{2}g\left(\frac{M_2}{b}\right)^2 + \frac{u}{b}\left(M_0 - b\cdot\frac{M_2}{b}\right)\ln\frac{M_0 - b\cdot\frac{M_2}{b}}{M_0} + u\cdot\frac{M_2}{b}$$

Cancel b and expand the first term.

$$y\left(t = \frac{M_2}{b}\right) = -\frac{1}{2}g\frac{M_2^2}{b^2} + \frac{u}{b}(M_0 - M_2)\ln\frac{M_0 - M_2}{M_0} + \frac{u}{b}M_2$$

Since $M_0 = M_1 + M_2$, we have $M_1 = M_0 - M_2$.

$$y\left(t = \frac{M_2}{b}\right) = -\frac{1}{2}g\frac{M_2^2}{b^2} + \frac{u}{b}M_1\ln\frac{M_1}{M_0} + \frac{u}{b}M_2$$

Factor u/b. Therefore,

$$y\left(t = \frac{M_2}{b}\right) = -\frac{1}{2}g\frac{M_2^2}{b^2} + \frac{u}{b}\left(M_1\ln\frac{M_1}{M_0} + M_2\right).$$

Part (d)

Velocity is defined as the rate at which the height increases.

$$v(t) = \frac{dy}{dt}$$

Multiply both sides by dt.

$$dy = v(t)\,dt$$

Integrate both sides.

$$\int dy = \int v(t)\,dt$$

Substitute the formula for $v(t)$ from part (b).

$$y = \int \left(-gt + u\ln\frac{M_0}{M_0 - bt}\right) dt$$

Split up the integral into two.

$$y = \int (-gt)\,dt + \int u\ln\frac{M_0}{M_0 - bt}\,dt$$

Bring the constants in front.

$$y = -g\int t\,dt + u\int \ln\frac{M_0}{M_0 - bt}\,dt$$

Evaluate the first integral and divide the numerator and denominator of the logarithm's argument by M_0.

$$y = -g\frac{t^2}{2} + u\int \ln\frac{1}{1 - \frac{bt}{M_0}}\,dt$$

Use a substitution to solve the last integral.

$$r = 1 - \frac{bt}{M_0}$$

$$dr = -\frac{b}{M_0}\,dt \quad \rightarrow \quad -\frac{M_0}{b}\,dr = dt$$

We get

$$y = -\frac{1}{2}gt^2 + u\int \ln\frac{1}{r}\left(-\frac{M_0}{b}\,dr\right).$$

Move the constants in front of the integral.

$$y = -\frac{1}{2}gt^2 - \frac{M_0 u}{b}\int \ln\frac{1}{r}\,dr$$

Invert the logarithm's argument and change the sign of the integral.

$$y = -\frac{1}{2}gt^2 + \frac{M_0 u}{b}\int \ln r\,dr$$

Use integration by parts.

$$q = \ln r \qquad ds = dr$$
$$dq = \frac{1}{r}\,dr \qquad s = r$$

We get

$$y = -\frac{1}{2}gt^2 + \frac{M_0 u}{b}\left(r\ln r - \int dr\right).$$

Evaluate the final integral.

$$y = -\frac{1}{2}gt^2 + \frac{M_0 u}{b}(r\ln r - r) + E$$

Factor r.

$$y = -\frac{1}{2}gt^2 + \frac{M_0 u}{b}r(\ln r - 1) + E$$

Change r back to t.

$$y(t) = -\frac{1}{2}gt^2 + \frac{M_0 u}{b}\left(1 - \frac{bt}{M_0}\right)\left[\ln\left(1 - \frac{bt}{M_0}\right) - 1\right] + E$$

If we assume the rocket launches from the ground, then the initial condition is $y(0) = 0$.

$$y(0) = \frac{M_0 u}{b}(-1) + E = 0 \quad \rightarrow \quad E = \frac{M_0 u}{b}$$

Plug this expression for E into the equation for $y(t)$.

$$y(t) = -\frac{1}{2}gt^2 + \frac{M_0 u}{b}\left(1 - \frac{bt}{M_0}\right)\left[\ln\left(1 - \frac{bt}{M_0}\right) - 1\right] + \frac{M_0 u}{b}$$

Distribute $M_0 u/b$.

$$y(t) = -\frac{1}{2}gt^2 + \left(\frac{M_0 u}{b} - ut\right)\left[\ln\left(1 - \frac{bt}{M_0}\right) - 1\right] + \frac{M_0 u}{b}$$

Distribute $M_0 u/b - ut$.

$$y(t) = -\frac{1}{2}gt^2 + \left(\frac{M_0 u}{b} - ut\right)\ln\left(1 - \frac{bt}{M_0}\right) - \frac{M_0 u}{b} + ut + \frac{M_0 u}{b}$$

Cancel $M_0 u/b$ and factor u/b.

$$y(t) = -\frac{1}{2}gt^2 + \frac{u}{b}(M_0 - bt)\ln\left(1 - \frac{bt}{M_0}\right) + ut$$

Write the logarithm's argument as one term. Therefore,

$$y(t) = -\frac{1}{2}gt^2 + \frac{u}{b}(M_0 - bt)\ln\frac{M_0 - bt}{M_0} + ut.$$