Problem 5

Find the curve $y = f(x)$ such that $f(x) \geq 0$, $f(0) = 0$, $f(1) = 1$, and the area under the graph of f from 0 to x is proportional to the $(n+1)$st power of $f(x)$.

Solution

$$\underbrace{\int_0^x f(t)\,dt}_{\text{The area under the graph of } f \text{ from 0 to } x} \qquad \underbrace{\propto}_{\text{is proportional to}} \qquad \underbrace{[f(x)]^{n+1}}_{\text{the } (n+1)\text{st power of } f(x)}$$

To change the proportionality to an equation, we must introduce a constant of proportionality, k.

$$\int_0^x f(t)\,dt = k[f(x)]^{n+1}$$

To remove the integral from the equation, differentiate both sides with respect to x and use the fundamental theorem of calculus, which says that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

$$\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = \frac{d}{dx}\left\{k[f(x)]^{n+1}\right\}$$

$$f(x) = k(n+1)[f(x)]^n \cdot f'(x)$$

Now separate variables.

$$k(n+1)f^n\frac{df}{dx} = f$$

$$f^{n-1}\,df = \frac{dx}{k(n+1)}$$

Integrate both sides.

$$\frac{f^n}{n} = \frac{x}{k(n+1)} + C$$

Now we can plug in the boundary conditions to determine the two constants.

$$f(0) = 0 \quad \rightarrow \quad \frac{0^n}{n} = \frac{0}{k(n+1)} + C \quad \rightarrow \quad C = 0$$

$$f(1) = 1 \quad \rightarrow \quad \frac{1^n}{n} = \frac{1}{k(n+1)} \quad \rightarrow \quad k = \frac{n}{n+1}$$

With these constants,

$$f^n = x.$$

Therefore,

$$f(x) = \sqrt[n]{x}.$$