Problem 9

A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:

(i) The rabbit is at the origin and the dog is at the point $(L, 0)$ at the instant the dog first sees the rabbit.
(ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.
(iii) The dog runs at the same speed as the rabbit.

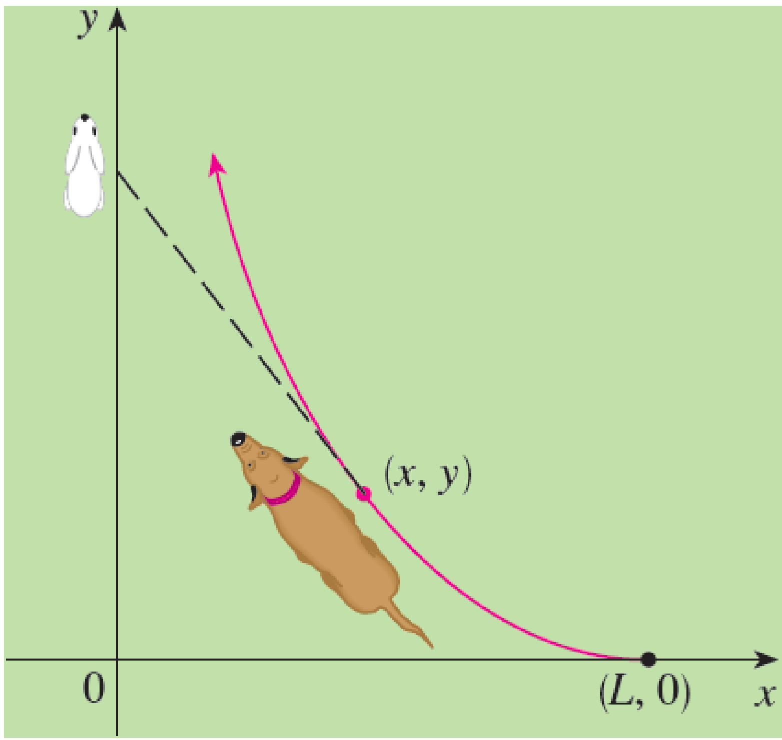


FIGURE FOR PROBLEM 9

(a) Show that the dog's path is the graph of the function $y = f(x)$, where y satisfies the differential equation

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(b) Determine the solution of the equation in part (a) that satisfies the initial conditions $y = y' = 0$ when $x = L$. [*Hint:* Let $z = dy/dx$ in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y.

(c) Does the dog ever catch the rabbit?

Solution

Part (a)

There is a critical observation that needs to be made in order to derive this formula. The dog and the rabbit travel at the same speed. Therefore, the arc length of the dog's path is equal to the length of the rabbit's path. In other words, the y-intercept of the tangent line to the dog's path at x is equal to the arc length of the dog's path. Recall from elementary algebra that if we have two points (x_1, y_1) and (x_2, y_2), the slope of the line passing through them is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The dog's position is (x, y), and the rabbit's position is $(0, y_{\text{rabbit}})$. But as said previously

$$y_{\text{rabbit}} = \text{dog's arc length from } x \text{ to } L = s|_x^L .$$

The slope, m, at x is dy/dx. Therefore,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y - s|_x^L}{x - 0} \\
x\frac{dy}{dx} &= y - s|_x^L \\
x\frac{dy}{dx} &= y - \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx
\end{aligned}$$

Now take the derivative of both sides to eliminate the integral.

$$\begin{aligned}
\frac{d}{dx}\left(x\frac{dy}{dx}\right) &= \frac{d}{dx}\left[y - \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx\right] \\
\cancel{\frac{dy}{dx}} + x\frac{d^2y}{dx^2} &= \cancel{\frac{dy}{dx}} - \frac{d}{dx}\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
x\frac{d^2y}{dx^2} &= -\frac{d}{dx}\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx
\end{aligned}$$

Here we have to use the fundamental theorem of calculus, which says that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Switch the limits and change the sign to get this form and apply the theorem.

$$x\frac{d^2y}{dx^2} = \frac{d}{dx}\int_L^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

And we arrive at the desired result,

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Part (b)

In order to solve this equation with the method of separation of variables, make use of the substitution given in the hint.

$$z = \frac{dy}{dx}$$

Although this equation is second-order, we can solve it by separation of variables because the equation is first-order in dy/dx.

$$x\frac{dz}{dx} = \sqrt{1+z^2}$$

We can solve for $z(x)$ by bringing all terms with z to the left and all constants and terms with x to the right and then integrating both sides.

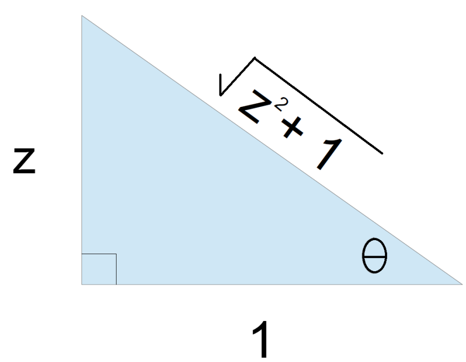
$$\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{dx}{x}$$

To integrate the left side, we will use a trigonometric substitution.

$$z = \tan\theta \quad \rightarrow \quad 1+z^2 = \sec^2\theta$$
$$dz = \sec^2\theta\, d\theta$$

$$\int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \ln x + C$$
$$\int \sec\theta\, d\theta = \ln x + C$$
$$\ln|\sec\theta + \tan\theta| = \ln x + C$$

To determine $\sec\theta$ and $\tan\theta$ in terms of z, draw the right triangle that is defined by the initial trigonometric substitution, $z = \tan\theta$.



$$\sec\theta = \sqrt{z^2+1} \quad \text{and} \quad \tan\theta = z.$$

So

$$
\begin{aligned}
\ln\left|\sqrt{z^2+1}+z\right| &= \ln x + C \\
e^{\ln\left|\sqrt{z^2+1}+z\right|} &= e^{\ln x + C} \\
\left|\sqrt{z^2+1}+z\right| &= e^C e^{\ln x} \\
\sqrt{z^2+1}+z &= \pm e^C x \\
\sqrt{z^2+1}+z &= Ax \\
\sqrt{z^2+1} &= Ax - z \\
z^2+1 &= A^2x^2 - 2Axz + z^2 \\
z &= \frac{Ax}{2} - \frac{1}{2Ax}
\end{aligned}
$$

Now that we solved for z, we can solve for y by simply integrating.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{Ax}{2} - \frac{1}{2Ax} \\
y &= \int \left(\frac{Ax}{2} - \frac{1}{2Ax}\right) dx \\
y(x) &= \frac{Ax^2}{4} - \frac{1}{2A}\ln x + D
\end{aligned}
$$

All that's left is to determine the constants of integration. We use the given boundary conditions, $y(L) = 0$ and $y'(L) = 0$.

$$
\begin{aligned}
y'(L) &= \frac{AL}{2} - \frac{1}{2AL} = 0 \quad \rightarrow \quad A = \frac{1}{L} \\
y(L) &= \frac{AL^2}{4} - \frac{1}{2A}\ln L + D \\
&= \frac{L}{4} - \frac{L}{2}\ln L + D = 0 \quad \rightarrow \quad D = \frac{L}{2}\ln L - \frac{L}{4}
\end{aligned}
$$

Therefore,

$$
y(x) = \frac{x^2}{4L} + \frac{L}{2}\ln\frac{L}{x} - \frac{L}{4}.
$$

The graph of this function is shown below for $L = 10$.

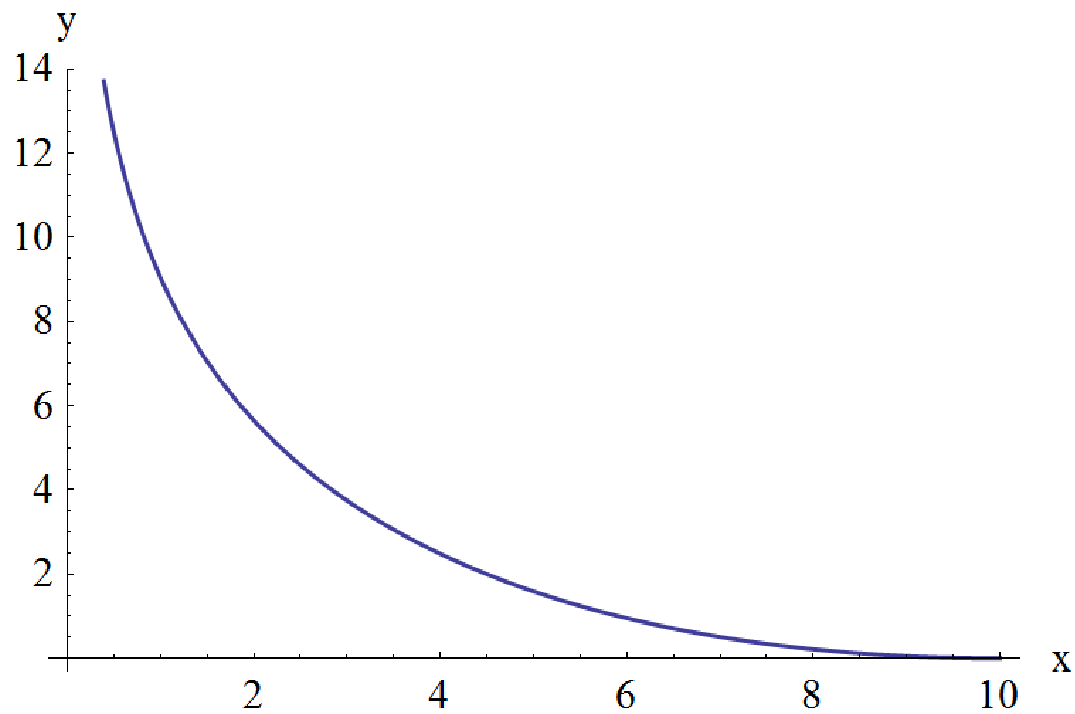


Figure 1: Plot of $y(x)$ vs. x for $L = 10$.

Part (c)

The dog will never catch the rabbit because they travel at the same speed.