Exercise 32

Find the orthogonal trajectories of the family of curves. Use a graphing device to draw several members of each family on a common screen.

$$y = \frac{1}{x+k}$$

Solution

To find the orthogonal trajectories, we have to solve for $y'(x)$, set $y'_\perp$ equal to the negative reciprocal, and then solve for $y_\perp$. Start by differentiating both sides of the given equation with respect to x.

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{x+k}\right)$$

$$\frac{dy}{dx} = -\frac{1}{(x+k)^2} \tag{1}$$

Solve the original equation for k. Multiply both sides by $x + k$.

$$y(x+k) = 1$$

Distribute y.

$$xy + ky = 1$$

Bring xy to the right side.

$$ky = 1 - xy$$

Divide both sides by y.

$$k = \frac{1 - xy}{y}$$

Plug the expression for k into equation (1).

$$\frac{dy}{dx} = -\frac{1}{\left(x + \frac{1-xy}{y}\right)^2}$$

Combine the two terms in the denominator.

$$\frac{dy}{dx} = -\frac{1}{\left(\frac{\cancel{xy}+1-\cancel{xy}}{y}\right)^2}$$

So we have

$$\frac{dy}{dx} = -y^2.$$

Here is where we introduce $y_\perp$.

$$\frac{dy_\perp}{dx} = \frac{1}{y_\perp^2}$$

Since this equation is separable, we can solve for $y_\perp$ by bringing all terms with $y_\perp$ to the left and all constants and terms with x to the right and then integrating both sides.

$$y_\perp^2\, dy_\perp = dx$$
$$\int y_\perp^2\, dy_\perp = \int dx$$
$$\frac{1}{3}y_\perp^3 = x + C$$

Multiply both sides by 3.

$$y_\perp^3 = 3x + 3C$$

Take the cube root of both sides. Let $A = 3C$.

$$y_\perp = \sqrt[3]{3x + A}$$

This is the family of curves orthogonal to $y = 1/(x + k)$.

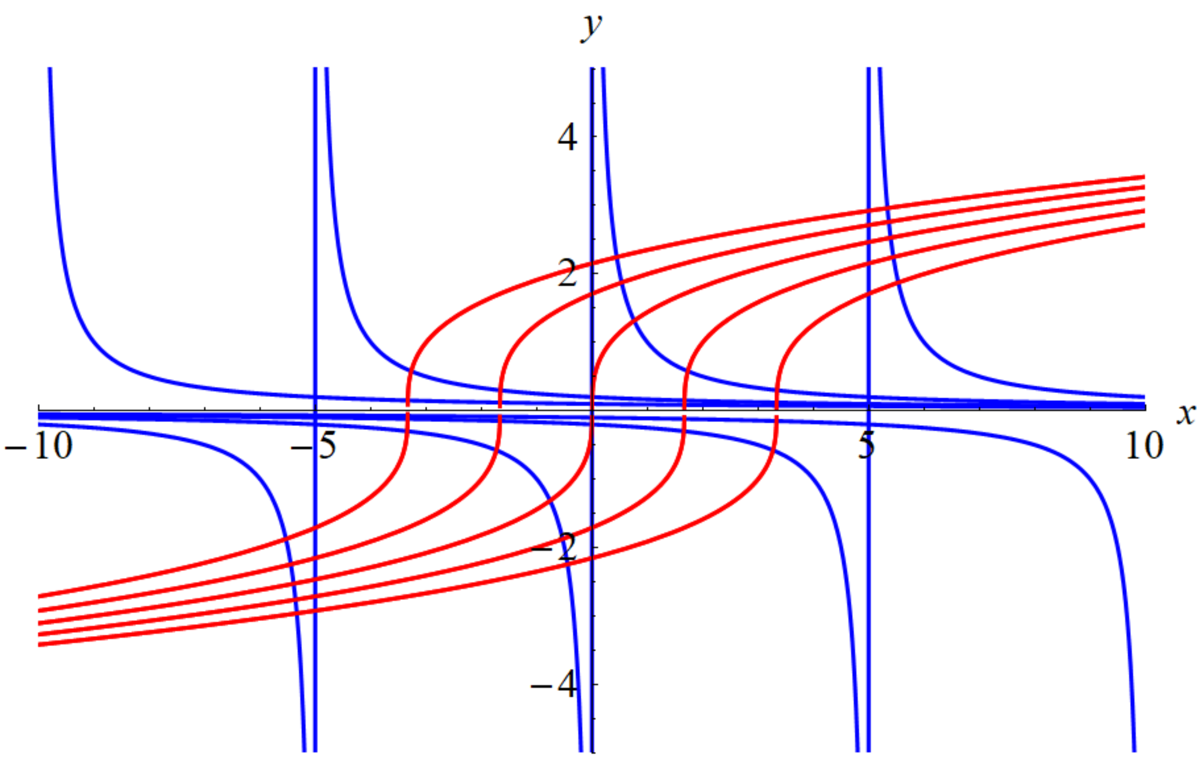

Figure 1: Plot of $y = 1/(x + k)$ in blue ($k = 0, \pm 5, \pm 10$) and the orthogonal trajectories $y_\perp$ in red ($A = 0, \pm 5, \pm 10$).