Exercise 33

An **integral equation** is an equation that contains an unknown function $y(x)$ and an integral that involves $y(x)$. Solve the given integral equation. [*Hint:* Use an initial condition obtained from the integral equation.]

$$y(x) = 2 + \int_2^x [t - ty(t)]\,dt$$

Solution

Right away we see that if we plug in $x = 2$ to the integral equation, we get

$$\begin{aligned} y(2) &= 2 + \int_2^2 [t - ty(t)]\,dt \\ &= 2 + 0 \\ &= 2, \end{aligned}$$

so the initial condition is $y(2) = 2$. In order to solve for $y(x)$ by the method introduced in this section, differentiate both sides of the integral equation with respect to x.

$$\frac{d}{dx}y(x) = \underbrace{\frac{d}{dx}2}_{=\,0} + \frac{d}{dx}\int_2^x [t - ty(t)]\,dt$$

By the fundamental theorem of calculus,

$$\frac{d}{dx}\int_2^x [t - ty(t)]\,dt = x - xy(x),$$

so the differential equation we need to solve is the following.

$$\frac{dy}{dx} = x - xy = x(1 - y)$$

This is a separable equation, which means we can solve for $y(x)$ by bringing all terms with y to the left and all constants and terms with x to the right and then integrating both sides.

$$\begin{aligned} dy &= x(1 - y)\,dx \\ \frac{dy}{1 - y} &= x\,dx \\ \int \frac{dy}{1 - y} &= \int x\,dx \end{aligned}$$

Use a u-substitution to solve the integral on the left.

$$\begin{aligned} \text{Let } u &= 1 - y \\ du &= -dy \quad \rightarrow \quad -du = dy \end{aligned}$$

$$\begin{aligned} \int \frac{-du}{u} &= \int x\,dx \\ \ln|u| &= -\frac{1}{2}x^2 + C \\ e^{\ln|u|} &= e^{-\frac{1}{2}x^2 + C} \\ |1 - y| &= e^{-\frac{1}{2}x^2}e^C \\ 1 - y &= \pm e^C e^{-\frac{1}{2}x^2} \end{aligned}$$

Let $C_1 = \pm e^C$. Then

$$y(x) = 1 - C_1 e^{-\frac{1}{2}x^2}.$$

Now we use the initial condition, $y(2) = 2$ to determine C_1.

$$\begin{aligned} y(2) = 1 - C_1 e^{-\frac{1}{2}2^2} &= 2 \\ 1 - C_1 e^{-2} &= 2 \\ -\frac{C_1}{e^2} &= 1 \\ C_1 &= -e^2 \end{aligned}$$

Therefore,

$$y(x) = 1 + e^2 e^{-\frac{1}{2}x^2}.$$