Exercise 36

Find a function f such that $f(3) = 2$ and

$$(t^2 + 1)f'(t) + [f(t)]^2 + 1 = 0 \qquad t \neq 1$$

[*Hint:* Use the addition formula for $\tan(x + y)$ on Reference Page 2.]

Solution

Solving for $f'(t)$ gives

$$f'(t) = -\frac{[f(t)]^2 + 1}{t^2 + 1}.$$

This differential equation is separable, which means we can solve for $f(t)$ by bringing the terms with f to the left and the constants and terms with t to the right and then integrating both sides.

$$\begin{aligned} \frac{df}{dt} &= -\frac{[f(t)]^2 + 1}{t^2 + 1} \\ df &= -\frac{[f(t)]^2 + 1}{t^2 + 1}\,dt \\ \frac{df}{[f(t)]^2 + 1} &= -\frac{dt}{t^2 + 1} \\ \int \frac{df}{[f(t)]^2 + 1} &= \int -\frac{dt}{t^2 + 1} \end{aligned}$$

Recall that

$$\int \frac{dx}{x^2 + 1} = \tan^{-1} x + C.$$

So

$$\begin{aligned} \tan^{-1} f &= -\tan^{-1} t + C \\ f(t) &= \tan\left(-\tan^{-1} t + C\right). \end{aligned}$$

Because $f(3) = 2$, we can figure out what C is.

$$\begin{aligned} f(3) = \tan\left(-\tan^{-1} 3 + C\right) &= 2 \\ -\tan^{-1} 3 + C &= \tan^{-1} 2 \\ C &= \tan^{-1} 2 + \tan^{-1} 3 \end{aligned}$$

Therefore,

$$f(t) = \tan\left(\tan^{-1} 2 + \tan^{-1} 3 - \tan^{-1} t\right).$$