Exercise 38

In Exercise 9.2.28 we discussed a differential equation that models the temperature of a 95°C cup of coffee in a 20°C room. Solve the differential equation to find an expression for the temperature of the coffee at time t.

Solution

Newton's law of cooling states that the rate of cooling of an object is proportional to the difference of the object's temperature and its surroundings. That is,

$$-\frac{dT}{dt} \propto T - T_{\text{surroundings}},$$

where $\propto$ means "proportional to." Note that the rate of cooling refers to how fast the temperature decreases with respect to time, so it is denoted as $-dT/dt$. In order to change $\propto$ to $=$, we must introduce a constant of proportionality, h.

$$-\frac{dT}{dt} = h(T - T_{\text{surroundings}})$$

This is the differential equation we have to solve. It is separable, so we solve for $T(t)$ by bringing all terms with T to the left and all constants and terms with t to the right and then integrating both sides.

$$\begin{aligned}
\frac{dT}{dt} &= -h(T-20) \\
dT &= -h(T-20)\,dt \\
\frac{dT}{T-20} &= -h\,dt \\
\int \frac{dT}{T-20} &= -\int h\,dt
\end{aligned}$$

Use a u-substitution to solve the integral on the left.

$$\begin{aligned}
\text{Let } u &= T-20 \\
du &= dT
\end{aligned}$$

$$\begin{aligned}
\int \frac{du}{u} &= -\int h\,dt \\
\ln|u| &= -ht + C \\
e^{\ln|T-20|} &= e^{-ht+C} \\
|T-20| &= e^{-ht}e^{C} \\
T-20 &= \pm e^{C}e^{-ht}
\end{aligned}$$

Let $C_1 = \pm e^C$. Then

$$T(t) = 20 + C_1 e^{-ht}.$$

We know that the initial temperature of the coffee is 95°C, so $T(0) = 95$. We can use this to determine C_1.

$$\begin{aligned} T(0) = 20 + C_1 &= 95 \\ C_1 &= 75 \end{aligned}$$

Therefore,

$$T(t) = 20 + 75e^{-ht}.$$