Exercise 41

In contrast to the situation of Exercise 40, experiments show that the reaction $H_2 + Br_2 \longrightarrow 2\,HBr$ satisfies the rate law

$$\frac{d[\text{HBr}]}{dt} = k[\text{H}_2][\text{Br}_2]^{1/2}$$

and so for this reaction the differential equation becomes

$$\frac{dx}{dt} = k(a-x)(b-x)^{1/2}$$

where $x = [\text{HBr}]$ and a and b are the initial concentrations of hydrogen and bromine.

(a) Find x as a function of t in the case where $a = b$. Use the fact that $x(0) = 0$.

(b) If $a > b$, find t as a function of x. [*Hint:* In performing the integration, make the substitution $u = \sqrt{b-x}$.]

Solution

Part (a)

In the case where $a = b$, the differential equation simplifies to

$$\frac{dx}{dt} = k(a-x)^{3/2}.$$

This equation is separable, so we can solve for $x(t)$ by bringing all terms with x to the left and all constants and terms with t to the right and then integrating both sides.

$$dx = k(a-x)^{3/2}\,dt$$

$$\frac{dx}{(a-x)^{3/2}} = k\,dt$$

$$\int \frac{dx}{(a-x)^{3/2}} = \int k\,dt$$

Use a u-substitution to solve the integral on the left.

$$\text{Let } u = a - x$$

$$du = -dx$$

$$\int \frac{-du}{u^{3/2}} = \int k\,dt$$

$$\frac{-1}{-\frac{1}{2}}\frac{1}{u^{1/2}} = kt + C$$

$$2\frac{1}{(a-x)^{1/2}} = kt + C$$

$$\frac{2}{kt+C} = (a-x)^{1/2}$$

$$\frac{4}{(kt+C)^2} = a - x$$

$$x(t) = a - \frac{4}{(kt+C)^2}$$

We're told that $x(0) = 0$, so we can determine C.

$$\begin{aligned} x(0) = a - \frac{4}{C^2} &= 0 \\ a &= \frac{4}{C^2} \\ C^2 &= \frac{4}{a} \\ C &= \pm\frac{2}{\sqrt{a}} \end{aligned}$$

We choose C to be $+\frac{2}{\sqrt{a}}$ so that the denominator doesn't equal 0 for any $t > 0$. Therefore,

$$x(t) = a - \frac{4}{\left(kt + \frac{2}{\sqrt{a}}\right)^2}, \quad a = b.$$

Part (b)

Now we assume that $a > b$, so the differential equation remains as given.

$$\frac{dx}{dt} = k(a - x)(b - x)^{1/2}$$

The equation is separable, so to solve for $x(t)$ we have to bring all terms with x to the left and all constants and terms with t to the right and then integrate both sides.

$$\begin{aligned} dx &= k(a - x)(b - x)^{1/2}\,dt \\ \frac{dx}{(a - x)(b - x)^{1/2}} &= k\,dt \\ \int \frac{dx}{(a - x)(b - x)^{1/2}} &= \int k\,dt \end{aligned}$$

Use the u-substition given in the hint to solve the integral on the left.

$$\begin{aligned} \text{Let } u = \sqrt{b - x} \quad &\rightarrow \quad u^2 = b - x \quad \rightarrow \quad u^2 - b + a = a - x \\ du = -\frac{1}{2\sqrt{b - x}}\,dx \quad &\rightarrow \quad -2\,du = \frac{1}{\sqrt{b - x}}\,dx \end{aligned}$$

$$\int \frac{-2\,du}{u^2 + (a - b)} = \int k\,dt$$

Recall that

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}.$$

So

$$\begin{aligned} \frac{-2}{\sqrt{a - b}}\tan^{-1}\frac{u}{\sqrt{a - b}} &= kt + C_1 \\ -\frac{2}{\sqrt{a - b}}\tan^{-1}\frac{\sqrt{b - x}}{\sqrt{a - b}} - C_1 &= kt \\ t(x) &= -\frac{1}{k}\left(\frac{2}{\sqrt{a - b}}\tan^{-1}\frac{\sqrt{b - x}}{\sqrt{a - b}} + C_1\right). \end{aligned}$$

If we use the initial condition, $x(0) = 0$, as we did before, we get

$$t(0) = -\frac{1}{k}\left(\frac{2}{\sqrt{a-b}}\tan^{-1}\frac{\sqrt{b}}{\sqrt{a-b}} + C_1\right) = 0$$

$$\frac{2}{\sqrt{a-b}}\tan^{-1}\frac{\sqrt{b}}{\sqrt{a-b}} + C_1 = 0$$

$$C_1 = -\frac{2}{\sqrt{a-b}}\tan^{-1}\frac{\sqrt{b}}{\sqrt{a-b}}.$$

Therefore,

$$t(x) = -\frac{1}{k}\left(\frac{2}{\sqrt{a-b}}\tan^{-1}\frac{\sqrt{b-x}}{\sqrt{a-b}} - \frac{2}{\sqrt{a-b}}\tan^{-1}\frac{\sqrt{b}}{\sqrt{a-b}}\right)$$

$$t(x) = -\frac{2}{k\sqrt{a-b}}\left(\tan^{-1}\frac{\sqrt{b-x}}{\sqrt{a-b}} - \tan^{-1}\frac{\sqrt{b}}{\sqrt{a-b}}\right)$$

$$t(x) = \frac{2}{k\sqrt{a-b}}\left(\tan^{-1}\frac{\sqrt{b}}{\sqrt{a-b}} - \tan^{-1}\frac{\sqrt{b-x}}{\sqrt{a-b}}\right), \quad a > b.$$