Exercise 43

A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.

(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.

(b) Assuming that $C_0 < r/k$, find $\lim_{t\to\infty} C(t)$ and interpret your answer.

Solution

Part (a)

The differential equation is separable, so we can solve for $C(t)$ by bringing all terms with x to the left and all constants and terms with t to the right and then integrating both sides.

$$\begin{aligned} \frac{dC}{dt} &= r - kC \\ dC &= (r - kC)\,dt \\ \frac{dC}{r - kC} &= dt \\ \int \frac{dC}{r - kC} &= \int dt \end{aligned}$$

Solve the integral on the left with a u-substitution.

$$\text{Let } u = r - kC$$

$$du = -k\,dC \quad \rightarrow \quad \frac{du}{-k} = dC$$

$$\begin{aligned} \int \frac{1}{-k}\frac{du}{u} &= \int dt \\ -\frac{1}{k}\ln|u| &= t + C_1 \\ \ln|r - kC| &= -kt - kC_1 \\ e^{\ln|r-kC|} &= e^{-kt-kC_1} \\ |r - kC| &= e^{-kC_1}e^{-kt} \\ r - kC &= \pm e^{-kC_1}e^{-kt} \end{aligned}$$

Let $C_2 = \pm e^{-kC_1}$.

$$r - kC = C_2 e^{-kt}$$

$$C(t) = \frac{1}{k}\left(r - C_2e^{-kt}\right)$$

Since the concentration at $t = 0$ is C_0, $C(t = 0) = C_0$. We can use this to determine C_2.

$$\begin{aligned} C(0) = \frac{1}{k}\left(r - C_2e^0\right) &= C_0 \\ r - C_2 &= kC_0 \\ C_2 &= r - kC_0 \end{aligned}$$

Therefore,

$$C(t) = \frac{1}{k}\left[r - (r - kC_0)e^{-kt}\right]$$

Part (b)

Assume now that $C_0 < \frac{r}{k}$, i.e. that $\frac{r}{k} - C_0 > 0$.

$$\begin{aligned} C(t) &= \frac{1}{k}\left[r - k\left(\frac{r}{k} - C_0\right)e^{-kt}\right] \\ C(t) &= \frac{r}{k} - \left(\frac{r}{k} - C_0\right)e^{-kt} \end{aligned}$$

Now we can find the limit as t goes to infinity.

$$\begin{aligned} \lim_{t\to\infty} C(t) &= \lim_{t\to\infty}\left[\frac{r}{k} - \left(\frac{r}{k} - C_0\right)e^{-kt}\right] \\ \lim_{t\to\infty} C(t) &= \lim_{t\to\infty}\frac{r}{k} - \lim_{t\to\infty}\left(\frac{r}{k} - C_0\right)e^{-kt} \\ \lim_{t\to\infty} C(t) &= \frac{r}{k} - \left(\frac{r}{k} - C_0\right)\underbrace{\lim_{t\to\infty} e^{-kt}}_{=0} \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} C(t) = \frac{r}{k}.$$

What this means is that the concentration of glucose after a long time does not depend on what the initial concentration was. Whatever it was, the concentration will eventually be equal to the ratio of how fast glucose enters to how fast it disappears.