Exercise 44

A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. Let $x = x(t)$ denote the amount of new currency in circulation at time t, with $x(0) = 0$.

(a) Formulate a mathematical model in the form of an initial-value problem that represents the "flow" of the new currency into circulation.

(b) Solve the initial-value problem found in part (a).

(c) How long will it take for the new bills to account for 90% of the currency in circulation?

Solution

When I first read the problem I incorrectly thought the amount of new currency would just be $x(t) = 50,000,000t$ since whatever money comes into the banks would get replaced. The old money would come into the banks, and the new money would just accumulate there. The amount of money in circulation would just dwindle to 0 eventually in this case. However, this is not what happens.

Rather, whenever old currency is replaced, the new currency gets distributed back into circulation so there is a mixture of old bills and new bills. At the start there is only old currency, so all $\$50,000,000$ gets replaced on the first day. But later on only a fraction of the $\$50,000,000$ will be old bills. As time goes on then, less old bills will come into the banks and hence less new bills will be created. Therefore, the rate at which new bills are created is 50,000,000 times the fraction of old bills (that is, the amount of old bills divided by the total amount of money) in circulation.

$$\frac{dx}{dt} = 50,000,000 \times \frac{x_{\text{old}}(t)}{10,000,000,000}$$

Since the amount of old bills plus new bills has to total 10,000,000,000, i.e. $x(t) + x_{\text{old}}(t) = 10,000,000,000$, we can write this rate as

$$\frac{dx}{dt} = 50,000,000 \times \frac{10,000,000,000 - x(t)}{10,000,000,000}.$$

When $x(t) = 0$, there are only old bills, and the amount of new bills increases by 50,000,000 as expected. When the number of new bills is close to 10,000,000,000, however, the number of old bills is really small, so the number of new bills will not increase by much. This differential equation is separable, so we can solve it by bringing the terms with x to the left and the constants and terms with t to the right and then integrating both sides.

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{200}(10^{10} - x) \\ \frac{dx}{10^{10} - x} &= \frac{1}{200}\,dt \\ \int \frac{dx}{10^{10} - x} &= \int \frac{1}{200}\,dt \end{aligned}$$

To solve the integral with respect to x, use a u substitution.

$$\text{Let } u = 10^{10} - x$$
$$du = -dx \quad \rightarrow \quad -du = dx$$

So then

$$\int \frac{-du}{u} = \int \frac{1}{200}\, dt$$
$$-\ln|u| = \frac{1}{200}t + C$$
$$\ln|10^{10} - x| = -\frac{1}{200}t - C$$
$$|10^{10} - x| = e^{-\frac{1}{200}t - C}$$
$$10^{10} - x = \pm e^{-C} e^{-\frac{1}{200}t}$$
$$-x = -10^{10} + Ae^{-\frac{1}{200}t},$$

where $A = \pm e^{-C}$.

$$x(t) = 10^{10} - Ae^{-\frac{1}{200}t}$$

Now we plug in the initial condition, $x(0) = 0$, to find A.

$$x(0) = 10^{10} - Ae^{0} = 0$$
$$A = 10^{10}$$

Therefore,

$$x(t) = 10^{10} - 10^{10} e^{-\frac{1}{200}t}$$
$$x(t) = 10^{10}\left(1 - e^{-\frac{1}{200}t}\right).$$

This function is plotted below as a function of time.

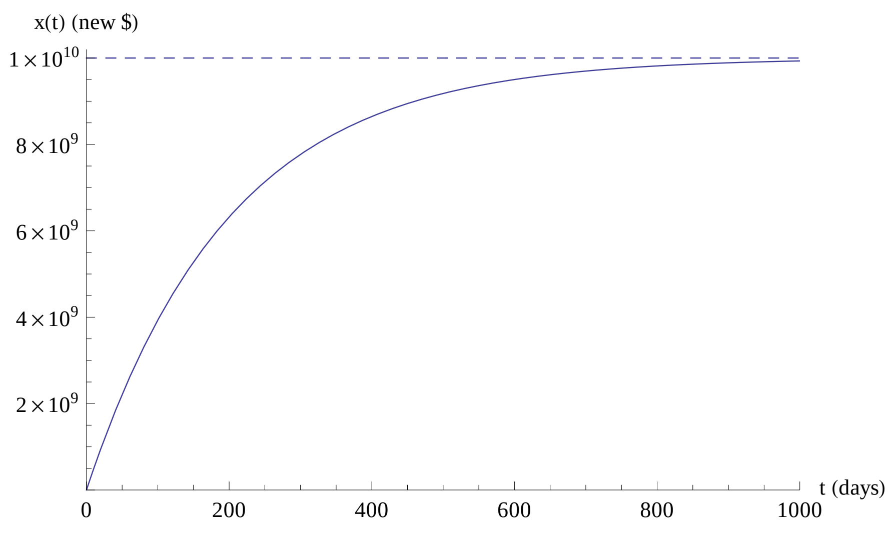

Figure 1: Plot of $x(t)$ vs. t.

To find how long it takes for the new currency to reach 90% of the total, we have to set $x(t) = 0.90 \times 10,000,000,000 = 9,000,000,000 = 9 \times 10^9$ and solve for t. That is,

$$\begin{aligned}
9 \times 10^9 &= 10 \times 10^9 \left(1 - e^{-\frac{1}{200}t}\right) \\
0.9 &= 1 - e^{-\frac{1}{200}t} \\
e^{-\frac{1}{200}t} &= 0.1 \\
\ln e^{-\frac{1}{200}t} &= \ln 0.1 \\
-\frac{1}{200}t &= \ln 0.1 = \ln 10^{-1} = -\ln 10 \\
t &= 200 \ln 10 \approx 460.5.
\end{aligned}$$

So it will take about 461 days for the new currency to reach 90% of the total amount in circulation.