Exercise 1

Verify the linearity and nonlinearity of the eight examples of PDEs given in the text, by checking whether or not equations (3) are valid.

Solution

The eight examples in the text are given below.

1.	$u_x + u_y = 0$	(transport)
2.	$u_x + yu_y = 0$	(transport)
3.	$u_x + uu_y = 0$	(shock wave)
4.	$u_{xx} + u_{yy} = 0$	(Laplace's equation)
5.	$u_{tt} - u_{xx} + u^3 = 0$	(wave with interaction)
6.	$u_t + uu_x + u_{xxx} = 0$	(dispersive wave)
7.	$u_{tt} + u_{xxxx} = 0$	(vibrating bar)
8.	$u_t - iu_{xx} = 0 \quad (i = \sqrt{-1})$	(quantum mechanics)

The equations in (3) are $\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v$ and $\mathscr{L}(cu) = c\mathscr{L}u$, which state the conditions for linearity. They will be checked in each case.

Example One:

$$\begin{aligned} u_x + u_y &= 0 \quad \text{(transport)} \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}u &= 0 \quad \text{(change notation)} \\ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u &= 0 \quad \text{(factor the common term)} \\ \mathscr{L}u &= 0 \end{aligned}$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$\begin{aligned} \mathscr{L}(u+v) &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(u+v) \\ &= \frac{\partial}{\partial x}(u+v) + \frac{\partial}{\partial y}(u+v) \\ &= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + \frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v \\ &= \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v + \frac{\partial}{\partial y}v \\ &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)v \\ &= \mathscr{L}u + \mathscr{L}v \end{aligned}$$

Thus, the first equation in (3) is satisfied. Now for the second equation in (3).

$$\begin{aligned}
\mathscr{L}(cu) &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(cu) \\
&= \frac{\partial}{\partial x}(cu) + \frac{\partial}{\partial y}(cu) \\
&= c\frac{\partial}{\partial x}u + c\frac{\partial}{\partial y}u \\
&= c\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}u\right) \\
&= c\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second equation in (3) is also satisfied, and hence the PDE in this example is linear.

Example Two:

$$\begin{aligned}
u_x + yu_y &= 0 \quad \text{(transport)} \\
\frac{\partial}{\partial x}u + y\frac{\partial}{\partial y}u &= 0 \quad \text{(change notation)} \\
\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u &= 0 \quad \text{(factor the common term)} \\
\mathscr{L}u &= 0
\end{aligned}$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(u+v) \\
&= \frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v) \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + y\frac{\partial}{\partial y}u + y\frac{\partial}{\partial y}v \\
&= \frac{\partial}{\partial x}u + y\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v + y\frac{\partial}{\partial y}v \\
&= \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

Thus, the first equation in (3) is satisfied. Now for the second equation in (3).

$$\begin{aligned}
\mathscr{L}(cu) &= \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(cu) \\
&= \frac{\partial}{\partial x}(cu) + y\frac{\partial}{\partial y}(cu) \\
&= c\frac{\partial}{\partial x}u + cy\frac{\partial}{\partial y}u \\
&= c\left(\frac{\partial}{\partial x}u + y\frac{\partial}{\partial y}u\right) \\
&= c\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second equation in (3) is also satisfied, and hence the PDE in this example is linear.

Example Three:

$$\begin{aligned}
u_x + uu_y &= 0 \quad \text{(shock wave)} \\
\frac{\partial}{\partial x}u + u\frac{\partial}{\partial y}u &= 0 \quad \text{(change notation)} \\
\left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)u &= 0 \quad \text{(factor the common term)} \\
\mathscr{L}u &= 0
\end{aligned}$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial}{\partial x} + u\frac{\partial}{\partial y}$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$\begin{aligned}
\mathscr{L}(u+v) &= \left[\frac{\partial}{\partial x} + (u+v)\frac{\partial}{\partial y}\right](u+v) \\
&= \frac{\partial}{\partial x}(u+v) + (u+v)\frac{\partial}{\partial y}(u+v) \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + (u+v)\left(\frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v\right) \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + u\frac{\partial}{\partial y}u + u\frac{\partial}{\partial y}v + v\frac{\partial}{\partial y}u + v\frac{\partial}{\partial y}v \\
&= \frac{\partial}{\partial x}u + u\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + u\frac{\partial}{\partial y}v + v\frac{\partial}{\partial y}u \\
&= \left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)v + u\frac{\partial}{\partial y}v + v\frac{\partial}{\partial y}u \\
&= \mathscr{L}u + \mathscr{L}v + uv_y + vu_y
\end{aligned}$$

Thus, the first equation in (3) is not satisfied. The PDE in this example is nonlinear.

Example Four:

$$\begin{aligned}
u_{xx} + u_{yy} &= 0 \quad \text{(Laplace's equation)} \\
\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u &= 0 \quad \text{(change notation)} \\
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u &= 0 \quad \text{(factor the common term)} \\
\mathscr{L}u &= 0
\end{aligned}$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u+v) \\
&= \frac{\partial^2}{\partial x^2}(u+v) + \frac{\partial^2}{\partial y^2}(u+v) \\
&= \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial y^2}v \\
&= \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

Thus, the first equation in (3) is satisfied. Now for the second equation in (3).

$$\begin{aligned}
\mathscr{L}(cu) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(cu) \\
&= \frac{\partial^2}{\partial x^2}(cu) + \frac{\partial^2}{\partial y^2}(cu) \\
&= c\frac{\partial^2}{\partial x^2}u + c\frac{\partial^2}{\partial y^2}u \\
&= c\left(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u\right) \\
&= c\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second equation in (3) is also satisfied, and hence the PDE in this example is linear.

Example Five:

$$
\begin{aligned}
u_{tt} - u_{xx} + u^3 &= 0 \quad \text{(wave with interaction)} \\
\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + u^3 &= 0 \quad \text{(change notation)} \\
\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + u^2\right) u &= 0 \quad \text{(factor the common term)} \\
\mathscr{L}u &= 0
\end{aligned}
$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + u^2$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$
\begin{aligned}
\mathscr{L}(u+v) &= \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + (u+v)^2\right](u+v) \\
&= \frac{\partial^2}{\partial t^2}(u+v) - \frac{\partial^2}{\partial x^2}(u+v) + (u+v)^3 \\
&= \frac{\partial^2}{\partial t^2}u + \frac{\partial^2}{\partial t^2}v - \frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial x^2}v + u^3 + u^2v + uv^2 + v^3 \\
&= \frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + u^3 + \frac{\partial^2}{\partial t^2}v - \frac{\partial^2}{\partial x^2}v + v^3 + u^2v + uv^2 \\
&= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + u^2\right)u + \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + v^2\right)v + u^2v + uv^2 \\
&= \mathscr{L}u + \mathscr{L}v + u^2v + uv^2
\end{aligned}
$$

Thus, the first equation in (3) is not satisfied. The PDE in this example is nonlinear.

Example Six:

$$
\begin{aligned}
u_t + uu_x + u_{xxx} &= 0 \quad \text{(dispersive wave)} \\
\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + \frac{\partial^3}{\partial x^3}u &= 0 \quad \text{(change notation)} \\
\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right)u &= 0 \quad \text{(factor the common term)} \\
\mathscr{L}u &= 0
\end{aligned}
$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$\begin{aligned}
\mathscr{L}(u+v) &= \left[\frac{\partial}{\partial t} + (u+v)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right](u+v) \\
&= \frac{\partial}{\partial t}(u+v) + (u+v)\frac{\partial}{\partial x}(u+v) + \frac{\partial^3}{\partial x^3}(u+v) \\
&= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v + (u+v)\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v\right) + \frac{\partial^3}{\partial x^3}u + \frac{\partial^3}{\partial x^3}v \\
&= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial x}u + v\frac{\partial}{\partial x}v + \frac{\partial^3}{\partial x^3}u + \frac{\partial^3}{\partial x^3}v \\
&= \frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + \frac{\partial^3}{\partial x^3}u + \frac{\partial}{\partial t}v + v\frac{\partial}{\partial x}v + \frac{\partial^3}{\partial x^3}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial x}u \\
&= \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right)u + \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right)v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial x}u \\
&= \mathscr{L}u + \mathscr{L}v + uv_x + vu_x
\end{aligned}$$

Thus, the first equation in (3) is not satisfied. The PDE in this example is nonlinear.

Example Seven:

$$\begin{aligned}
u_{tt} + u_{xxxx} &= 0 \quad \text{(vibrating bar)} \\
\frac{\partial^2}{\partial t^2}u + \frac{\partial^4}{\partial x^4}u &= 0 \quad \text{(change notation)} \\
\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right)u &= 0 \quad \text{(factor the common term)} \\
\mathscr{L}u &= 0
\end{aligned}$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right)(u+v) \\
&= \frac{\partial^2}{\partial t^2}(u+v) + \frac{\partial^4}{\partial x^4}(u+v) \\
&= \frac{\partial^2}{\partial t^2}u + \frac{\partial^2}{\partial t^2}v + \frac{\partial^4}{\partial x^4}u + \frac{\partial^4}{\partial x^4}v \\
&= \frac{\partial^2}{\partial t^2}u + \frac{\partial^4}{\partial x^4}u + \frac{\partial^2}{\partial t^2}v + \frac{\partial^4}{\partial x^4}v \\
&= \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right)u + \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

Thus, the first equation in (3) is satisfied. Now for the second equation in (3).

$$\begin{aligned}
\mathscr{L}(cu) &= \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right)(cu) \\
&= \frac{\partial^2}{\partial t^2}(cu) + \frac{\partial^4}{\partial x^4}(cu) \\
&= c\frac{\partial^2}{\partial t^2}u + c\frac{\partial^4}{\partial x^4}u \\
&= c\left(\frac{\partial^2}{\partial t^2}u + \frac{\partial^4}{\partial x^4}u\right) \\
&= c\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second equation in (3) is also satisfied, and hence the PDE in this example is linear.

Example Eight:

$$\begin{aligned}
u_t - iu_{xx} &= 0 \quad (i = \sqrt{-1}) \quad \text{(transport)} \\
\frac{\partial}{\partial t}u - i\frac{\partial^2}{\partial x^2}u &= 0 \quad \text{(change notation)} \\
\left(\frac{\partial}{\partial t} - i\frac{\partial^2}{\partial x^2}\right)u &= 0 \quad \text{(factor the common term)} \\
\mathscr{L}u &= 0
\end{aligned}$$

Therefore, for this example, the operator we are working with is

$$\mathscr{L} = \frac{\partial}{\partial t} - i\frac{\partial^2}{\partial x^2}$$

Now the equations in (3) may be verified. Suppose u and v are solutions to the PDE. The first equation in (3) yields

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial}{\partial t} - i\frac{\partial^2}{\partial x^2}\right)(u+v) \\
&= \frac{\partial}{\partial t}(u+v) - i\frac{\partial^2}{\partial x^2}(u+v) \\
&= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v - i\frac{\partial^2}{\partial x^2}u - i\frac{\partial^2}{\partial x^2}v \\
&= \frac{\partial}{\partial t}u - i\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial t}v - i\frac{\partial^2}{\partial x^2}v \\
&= \left(\frac{\partial}{\partial t} - i\frac{\partial^2}{\partial x^2}\right)u + \left(\frac{\partial}{\partial t} - i\frac{\partial^2}{\partial x^2}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

Thus, the first equation in (3) is satisfied. Now for the second equation in (3).

$$\begin{aligned}
\mathscr{L}(cu) &= \left(\frac{\partial}{\partial t} - i\frac{\partial^2}{\partial x^2}\right)(cu) \\
&= \frac{\partial}{\partial t}(cu) - i\frac{\partial^2}{\partial x^2}(cu) \\
&= c\frac{\partial}{\partial t}u - ic\frac{\partial^2}{\partial x^2}u \\
&= c\left(\frac{\partial}{\partial t}u - i\frac{\partial^2}{\partial x^2}u\right) \\
&= c\left(\frac{\partial}{\partial t} - i\frac{\partial^2}{\partial x^2}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second equation in (3) is also satisfied, and hence the PDE in this example is linear.