Exercise 3

For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

(a) $u_t - u_{xx} + 1 = 0$

(b) $u_t - u_{xx} + xu = 0$

(c) $u_t - u_{xxt} + uu_x = 0$

(d) $u_{tt} - u_{xx} + x^2 = 0$

(e) $iu_t - u_{xx} + u/x = 0$

(f) $u_x(1 + u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2} = 0$

(g) $u_x + e^y u_y = 0$

(h) $u_t + u_{xxxx} + \sqrt{1 + u} = 0$

Solution

The order of an equation is the highest derivative that appears. To determine if an operator is linear, one must check whether the conditions for linearity hold:

$$\mathscr{L}(u + v) = \mathscr{L}u + \mathscr{L}v \quad \text{and} \quad \mathscr{L}(cu) = c\mathscr{L}u$$

Assuming that $\mathscr{L}$ is a linear operator, the equation $\mathscr{L}u = 0$ is a homogeneous linear equation, and the equation $\mathscr{L}u = g$ $(g \neq 0)$ is an inhomogeneous linear equation.

Part (a)

$$u_t - u_{xx} + 1 = 0$$

This PDE is of the second order because the second derivative is the highest derivative.

The equation can be written as

$$\mathscr{L}u = -1, \qquad \text{where } \mathscr{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

Checking the first condition,

$$\begin{aligned}
\mathscr{L}(u + v) &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(u + v) \\
&= \frac{\partial}{\partial t}(u + v) - \frac{\partial^2}{\partial x^2}(u + v) \\
&= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v - \frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial x^2}v \\
&= \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial t}v - \frac{\partial^2}{\partial x^2}v \\
&= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)u + \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

The first condition for linearity holds. Now the second one must be checked.

$$\begin{aligned}\mathscr{L}(cu) &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(cu) \\ &= \frac{\partial}{\partial t}(cu) - \frac{\partial^2}{\partial x^2}(cu) \\ &= c\frac{\partial}{\partial t}u - c\frac{\partial^2}{\partial x^2}u \\ &= c\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)u \\ &= c\mathscr{L}u\end{aligned}$$

The second condition for linearity is satisfied as well, so the PDE is a linear inhomogeneous one.

Part (b)

$$u_t - u_{xx} + xu = 0$$

This PDE is of the second order because the second derivative is the highest derivative.

The equation can be written as

$$\mathscr{L}u = 0, \qquad \text{where } \mathscr{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x$$

Checking the first condition,

$$\begin{aligned}\mathscr{L}(u+v) &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x\right)(u+v) \\ &= \frac{\partial}{\partial t}(u+v) - \frac{\partial^2}{\partial x^2}(u+v) + x(u+v) \\ &= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v - \frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial x^2}v + xu + xv \\ &= \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u + xu + \frac{\partial}{\partial t}v - \frac{\partial^2}{\partial x^2}v + xv \\ &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x\right)u + \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x\right)v \\ &= \mathscr{L}u + \mathscr{L}v\end{aligned}$$

The first condition for linearity holds. Now the second one must be checked.

$$\begin{aligned}\mathscr{L}(cu) &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x\right)(cu) \\ &= \frac{\partial}{\partial t}(cu) - \frac{\partial^2}{\partial x^2}(cu) + x(cu) \\ &= c\frac{\partial}{\partial t}u - c\frac{\partial^2}{\partial x^2}u + cxu \\ &= c\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x\right)u \\ &= c\mathscr{L}u\end{aligned}$$

The second condition for linearity is satisfied as well, so the PDE is a linear homogeneous one.

Part (c)

$$u_t - u_{xxt} + uu_x = 0$$

This PDE is of the third order because the third derivative is the highest derivative.

The equation can be written as

$$\mathscr{L}u = 0, \qquad \text{where } \mathscr{L} = \frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^2 \partial t} + u\frac{\partial}{\partial x}$$

Checking the first condition,

$$\begin{aligned}
\mathscr{L}(u+v) &= \left[\frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^2 \partial t} + (u+v)\frac{\partial}{\partial x}\right](u+v) \\
&= \frac{\partial}{\partial t}(u+v) - \frac{\partial^3}{\partial x^2 \partial t}(u+v) + (u+v)\frac{\partial}{\partial x}(u+v) \\
&= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v - \frac{\partial^3}{\partial x^2 \partial t}u - \frac{\partial^3}{\partial x^2 \partial t}v + (u+v)\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v\right) \\
&= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v - \frac{\partial^3}{\partial x^2 \partial t}u - \frac{\partial^3}{\partial x^2 \partial t}v + u\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial x}u + v\frac{\partial}{\partial x}v \\
&= \frac{\partial}{\partial t}u - \frac{\partial^3}{\partial x^2 \partial t}u + u\frac{\partial}{\partial x}u + \frac{\partial}{\partial t}v - \frac{\partial^3}{\partial x^2 \partial t}v + v\frac{\partial}{\partial x}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial x}u \\
&= \left(\frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^2 \partial t} + u\frac{\partial}{\partial x}\right)u + \left(\frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^2 \partial t} + v\frac{\partial}{\partial x}\right)v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial x}u \\
&= \mathscr{L}u + \mathscr{L}v + uv_x + vu_x
\end{aligned}$$

The first condition for linearity does not hold, so the equation is nonlinear.

Part (d)

$$u_{tt} - u_{xx} + x^2 = 0$$

This PDE is of the second order because the second derivative is the highest derivative.

The equation can be written as

$$\mathscr{L}u = -x^2, \qquad \text{where } \mathscr{L} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

Checking the first condition,

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)(u+v) \\
&= \frac{\partial^2}{\partial t^2}(u+v) - \frac{\partial^2}{\partial x^2}(u+v) \\
&= \frac{\partial^2}{\partial t^2}u + \frac{\partial^2}{\partial t^2}v - \frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial x^2}v \\
&= \frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial t^2}v - \frac{\partial^2}{\partial x^2}v \\
&= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u + \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

The first condition for linearity holds. Now the second one must be checked.

$$\begin{aligned}
\mathscr{L}(cu) &= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)(cu) \\
&= \frac{\partial^2}{\partial t^2}(cu) - \frac{\partial^2}{\partial x^2}(cu) \\
&= c\frac{\partial^2}{\partial t^2}u - c\frac{\partial^2}{\partial x^2}u \\
&= c\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second condition for linearity is satisfied as well, so the PDE is a linear inhomogeneous one.

Part (e)

$$iu_t - u_{xx} + u/x = 0$$

This PDE is of the second order because the second derivative is the highest derivative.

The equation can be written as

$$\mathscr{L}u = 0, \qquad \text{where } \mathscr{L} = i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x}$$

Checking the first condition,

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x}\right)(u+v) \\
&= i\frac{\partial}{\partial t}(u+v) - \frac{\partial^2}{\partial x^2}(u+v) + \frac{1}{x}(u+v) \\
&= i\frac{\partial}{\partial t}u + i\frac{\partial}{\partial t}v - \frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial x^2}v + \frac{1}{x}u + \frac{1}{x}v \\
&= i\frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u + \frac{1}{x}u + i\frac{\partial}{\partial t}v - \frac{\partial^2}{\partial x^2}v + \frac{1}{x}v \\
&= \left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x}\right)u + \left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

The first condition for linearity holds. Now the second one must be checked.

$$\begin{aligned}
\mathscr{L}(cu) &= \left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x}\right)(cu) \\
&= i\frac{\partial}{\partial t}(cu) - \frac{\partial^2}{\partial x^2}(cu) + \frac{1}{x}(cu) \\
&= ic\frac{\partial}{\partial t}u - c\frac{\partial^2}{\partial x^2}u + c\frac{1}{x}u \\
&= c\left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second condition for linearity is satisfied as well, so the PDE is a linear homogeneous one.

Part (f)

$$u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$$

This PDE is of the first order because the first derivative is the highest derivative.

The equation can be written as

$$\begin{aligned}
\frac{1}{\sqrt{1+\left(\frac{\partial}{\partial x}u\right)^2}}\frac{\partial}{\partial x}u + \frac{1}{\sqrt{1+\left(\frac{\partial}{\partial y}u\right)^2}}\frac{\partial}{\partial y}u &= 0 \\
\left[\frac{1}{\sqrt{1+\left(\frac{\partial}{\partial x}u\right)^2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{1+\left(\frac{\partial}{\partial y}u\right)^2}}\frac{\partial}{\partial y}\right]u &= 0 \\
\mathscr{L}u &= 0
\end{aligned}$$

The operator for this PDE is

$$\mathscr{L} = \frac{1}{\sqrt{1+\left(\frac{\partial}{\partial x}u\right)^2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{1+\left(\frac{\partial}{\partial y}u\right)^2}}\frac{\partial}{\partial y}$$

Checking the first condition,

$$\begin{aligned}\mathscr{L}(u+v) &= \left[\frac{1}{\sqrt{1+\left[\frac{\partial}{\partial x}(u+v)\right]^2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{1+\left[\frac{\partial}{\partial y}(u+v)\right]^2}}\frac{\partial}{\partial y}\right](u+v) \\ &= \frac{1}{\sqrt{1+\left(\frac{\partial}{\partial x}u+\frac{\partial}{\partial x}v\right)^2}}\frac{\partial}{\partial x}(u+v) + \frac{1}{\sqrt{1+\left(\frac{\partial}{\partial y}u+\frac{\partial}{\partial y}v\right)^2}}\frac{\partial}{\partial y}(u+v)\end{aligned}$$

The first condition cannot be satisfied because the square roots cannot be simplified (one for u and one for v). Hence, the PDE is nonlinear.

Part (g)

$$u_x + e^y u_y = 0$$

This PDE is of the first order because the first derivative is the highest derivative.

The equation can be written as

$$\mathscr{L}u = 0, \qquad \text{where } \mathscr{L} = \frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y}$$

Checking the first condition,

$$\begin{aligned}\mathscr{L}(u+v) &= \left(\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y}\right)(u+v) \\ &= \frac{\partial}{\partial x}(u+v) + e^y\frac{\partial}{\partial y}(u+v) \\ &= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + e^y\frac{\partial}{\partial y}u + e^y\frac{\partial}{\partial y}v \\ &= \frac{\partial}{\partial x}u + e^y\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v + e^y\frac{\partial}{\partial y}v \\ &= \left(\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y}\right)v \\ &= \mathscr{L}u + \mathscr{L}v\end{aligned}$$

The first condition for linearity holds. Now the second one must be checked.

$$\begin{aligned}\mathscr{L}(cu) &= \left(\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y}\right)(cu) \\ &= \frac{\partial}{\partial x}(cu) + e^y\frac{\partial}{\partial y}(cu) \\ &= c\frac{\partial}{\partial x}u + ce^y\frac{\partial}{\partial y}u \\ &= c\left(\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y}\right)u \\ &= c\mathscr{L}u\end{aligned}$$

The second condition for linearity is satisfied as well, so the PDE is a linear homogeneous one.

Part (h)

$$u_t + u_{xxxx} + \sqrt{1+u} = 0$$

This PDE is of the fourth order because the fourth derivative is the highest derivative.

The equation can be written as

$$\mathscr{L}u = 0, \qquad \text{where } \mathscr{L} = \frac{\partial}{\partial t} + \frac{\partial^4}{\partial x^4} + \frac{\sqrt{1+u}}{u}$$

Checking the first condition,

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial}{\partial t} + \frac{\partial^4}{\partial x^4} + \frac{\sqrt{1+(u+v)}}{u+v}\right)(u+v) \\
&= \frac{\partial}{\partial t}(u+v) + \frac{\partial^4}{\partial x^4}(u+v) + \frac{\sqrt{1+u+v}}{u+v}(u+v) \\
&= \frac{\partial}{\partial t}u + \frac{\partial}{\partial t}v + \frac{\partial^4}{\partial x^4}u + \frac{\partial^4}{\partial x^4}v + \sqrt{1+u+v}
\end{aligned}$$

The first condition cannot be satisfied because the square root cannot be simplified (one for u and one for v). Hence, the PDE is nonlinear.