Exercise 5

Which of the following collections of 3-vectors \([a, b, c]\) are vector spaces? Provide reasons.

(a) The vectors with \(b = 0\).
(b) The vectors with \(b = 1\).
(c) The vectors with \(ab = 0\).
(d) All the linear combinations of the two vectors \([1, 1, 0]\) and \([2, 0, 1]\).
(e) All the vectors such that \(c - a = 2b\).

Solution

For a collection of 3-vectors \(V\) to be a vector space over \(\mathbb{R}\), the vector addition and scalar multiplication operations must satisfy the following ten properties:

(A1) \(x + y \in V\) for all \(x, y \in V\).

(A2) \((x + y) + z = x + (y + z)\) for every \(x, y, z \in V\).

(A3) \(x + y = y + x\) for every \(x, y \in V\).

(A4) There is an element \(0 \in V\) such that \(x + 0 = x\) for every \(x \in V\).

(A5) For each \(x \in V\), there is an element \((-x) \in V\) such that \(x + (-x) = 0\).

(M1) \(\alpha x \in V\) for all \(\alpha \in \mathbb{R}\) and \(x \in V\).

(M2) \((\alpha \beta)x = \alpha(\beta x)\) for all \(\alpha, \beta \in \mathbb{R}\) and every \(x \in V\).

(M3) \(\alpha(x + y) = \alpha x + \alpha y\) for every \(\alpha \in \mathbb{R}\) and all \(x, y \in V\).

(M4) \((\alpha + \beta)x = \alpha x + \beta x\) for all \(\alpha, \beta \in \mathbb{R}\) and every \(x \in V\).

(M5) \(1x = x\) for every \(x \in V\).

Part (a)

\[ V = \{[a, 0, c] \mid a, c \in \mathbb{R}\} \]

Choose \(x = [a_1, 0, c_1] \in V\) and \(y = [a_2, 0, c_2] \in V\) and \(z = [a_3, 0, c_3] \in V\).

\[ x + y = [a_1, 0, c_1] + [a_2, 0, c_2] = [a_1 + a_2, 0, c_1 + c_2] \]

Because \(a_1 + a_2 \in \mathbb{R}\) and \(c_1 + c_2 \in \mathbb{R}\), \(x + y \in V\). Hence, property A1 is satisfied.

\[
\begin{align*}
(x + y) + z &= ([a_1, 0, c_1] + [a_2, 0, c_2]) + [a_3, 0, c_3] \\
&= [a_1 + a_2, 0, c_1 + c_2] + [a_3, 0, c_3] \\
&= [a_1 + a_2 + a_3, 0, c_1 + c_2 + c_3] \\
&= [a_1, 0, c_1] + [a_2 + a_3, 0, c_2 + c_3] \\
&= [a_1, 0, c_1] + ([a_2, 0, c_2] + [a_3, 0, c_3]) \\
&= x + (y + z)
\end{align*}
\]
Property A2 is satisfied.

\[ x + y = [a_1, 0, c_1] + [a_2, 0, c_2] \]
\[ = [a_1 + a_2, 0, c_1 + c_2] \]
\[ = [a_2 + a_1, 0, c_2 + c_1] \]
\[ = [a_2, 0, c_2] + [a_1, 0, c_1] \]
\[ = y + x \]

Property A3 is satisfied. Setting \( a = 0 \) and \( c = 0 \), we find that \( 0 = [0, 0, 0] \in V \), so property A4 is satisfied. Since \( -a_1 \) and \( -c_1 \) are members of \( \mathbb{R} \), \( -x \in V \), and property A5 is satisfied. Choose \( \alpha_1 \in \mathbb{R} \) and \( \beta_1 \in \mathbb{R} \).

\[ \alpha_1 x = \alpha_1 [a_1, 0, c_1] \]
\[ = [\alpha_1 a_1, 0, \alpha_1 c_1] \]

Since \( \alpha_1 a_1 \) and \( \alpha_1 c_1 \) are members of \( \mathbb{R} \), \( \alpha_1 x \in V \), and property M1 is satisfied.

\[ (\alpha_1 \beta_1) x = (\alpha_1 \beta_1)[a_1, 0, c_1] \]
\[ = [\alpha_1 \beta_1 a_1, 0, \alpha_1 \beta_1 c_1] \]
\[ = \alpha_1 [\beta_1 a_1, 0, \beta_1 c_1] \]
\[ = \alpha_1 (\beta_1 x) \]

So property M2 is satisfied.

\[ \alpha_1 (x + y) = \alpha_1 ([a_1, 0, c_1] + [a_2, 0, c_2]) \]
\[ = \alpha_1 [a_1 + a_2, 0, c_1 + c_2] \]
\[ = [\alpha_1 (a_1 + a_2), 0, \alpha_1 (c_1 + c_2)] \]
\[ = [\alpha_1 a_1 + \alpha_1 a_2, 0, \alpha_1 c_1 + \alpha_1 c_2] \]
\[ = [\alpha_1 a_1, 0, \alpha_1 c_1] + [\alpha_1 a_2, 0, \alpha_1 c_2] \]
\[ = \alpha_1 [a_1, 0, c_1] + \alpha_1 [a_2, 0, c_2] \]
\[ = \alpha_1 x + \alpha_1 y \]

So property M3 is satisfied.

\[ (\alpha_1 + \beta_1) x = (\alpha_1 + \beta_1)[a_1, 0, c_1] \]
\[ = [(\alpha_1 + \beta_1) a_1, 0, (\alpha_1 + \beta_1) c_1] \]
\[ = [\alpha_1 a_1 + \beta_1 a_1, 0, \alpha_1 c_1 + \beta_1 c_1] \]
\[ = [\alpha_1 a_1, 0, \alpha_1 c_1] + [\beta_1 a_1, 0, \beta_1 c_1] \]
\[ = \alpha_1 [a_1, 0, c_1] + \beta_1 [a_1, 0, c_1] \]
\[ = \alpha_1 x + \beta_1 x \]

So property M4 is satisfied. \( 1x = 1[a_1, 0, c_1] = [1 \times a_1, 1 \times 0, 1 \times c_1] = [a_1, 0, c_1] = x \), so property M5 is satisfied.

All ten properties are satisfied, so \( V = \{ [a, 0, c] \mid a, c \in \mathbb{R} \} \) is a vector space.

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Part (b)

\[ \mathcal{V} = \{ [a, b, c] : a, b, c \in \mathbb{R} \} \]

There is no \( \mathbf{0} = [0, 0, 0] \) vector in this set of 3-vectors, so property A4 is not satisfied. Therefore, \( \mathcal{V} = \{ [a, 1, c] : a, c \in \mathbb{R} \} \) is not a vector space.

Part (c)

\[ \mathcal{V} = \{ [a, b, c] : ab = 0 \text{ and } a, b, c \in \mathbb{R} \} \]

Choose \( \mathbf{x} = [a_1, b_1, c_1] \in \mathcal{V} \) and \( \mathbf{y} = [a_2, b_2, c_2] \in \mathcal{V} \) and \( \mathbf{z} = [a_3, b_3, c_3] \in \mathcal{V} \).

\[ \mathbf{x} + \mathbf{y} = [a_1 + a_2, b_1 + b_2, c_1 + c_2] \]

The conditions \( a_1 b_1 = 0 \) and \( a_2 b_2 = 0 \) do not guarantee that \( (a_1 + a_2)(b_1 + b_2) = 0 \); for example, take \( a_1 = 0, b_1 = 1, a_2 = 1, \) and \( b_2 = 0 \). Therefore, \( \mathbf{x} + \mathbf{y} \notin \mathcal{V} \), and \( \mathcal{V} = \{ [a, b, c] : ab = 0 \text{ and } a, b, c \in \mathbb{R} \} \) is not a vector space.

Part (d)

\[ \mathcal{V} = \{ [a, b, c] : m[1, 1, 0] + n[2, 0, 1] = [a, b, c] = [m + 2n, m, n] : m, n \in \mathbb{R} \} \]

Choose \( \mathbf{x} = [m_1 + 2n_1, m_1, n_1] \in \mathcal{V} \) and \( \mathbf{y} = [m_2 + 2n_2, m_2, n_2] \in \mathcal{V} \) and \( \mathbf{z} = [m_3 + 2n_3, m_3, n_3] \in \mathcal{V} \).

\[ \mathbf{x} + \mathbf{y} = [m_1 + 2n_1 + m_2 + 2n_2, m_1 + m_2, n_1 + n_2] \]
\[ = [(m_1 + m_2) + 2(n_1 + n_2), m_1 + m_2, n_1 + n_2] \]
\[ = [m_4 + 2n_4, m_4, n_4] \]

Because \( m_4 \in \mathbb{R} \) and \( n_4 \in \mathbb{R} \), \( \mathbf{x} + \mathbf{y} \in \mathcal{V} \). Hence, property A1 is satisfied.

\[ (\mathbf{x} + \mathbf{y}) + \mathbf{z} = ([m_1 + 2n_1 + m_1, n_1] + [m_2 + 2n_2, m_2, n_2]) + [m_3 + 2n_3, m_3, n_3] \]
\[ = [m_1 + 2n_1 + m_2 + 2n_2 + m_1 + m_2, n_1 + n_2 + n_3] \]
\[ = [m_1 + 2n_1 + m_2 + 2n_2 + m_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3] \]
\[ = [m_1 + 2n_1, m_1, n_1] + ([m_2 + 2n_2, m_2, n_2] + [m_3 + 2n_3, m_3, n_3]) \]
\[ = \mathbf{x} + (\mathbf{y} + \mathbf{z}) \]

Property A2 is satisfied.

\[ \mathbf{x} + \mathbf{y} = [m_1 + 2n_1, m_1, n_1] + [m_2 + 2n_2, m_2, n_2] \]
\[ = [m_1 + 2n_1 + m_2 + 2n_2, m_1 + m_2, n_1 + n_2] \]
\[ = [m_2 + 2n_2 + m_1 + 2n_1, m_2 + m_1, n_2 + n_1] \]
\[ = [m_2 + 2n_2, m_2, n_2] + [m_1 + 2n_1, m_1, n_1] \]
\[ = \mathbf{y} + \mathbf{x} \]
Property A3 is satisfied. Setting \( m = 0 \) and \( n = 0 \), we find that \( 0 = [0,0,0] \in \mathcal{V} \), so property A4 is satisfied. Since \(-m_1\) and \(-n_1\) and \(-(m_1 + 2n_1)\) are members of \( \mathbb{R} \), \(-x \in \mathcal{V} \), and property A5 is satisfied.

Choose \( \alpha_1 \in \mathbb{R} \) and \( \beta_1 \in \mathbb{R} \).

\[
\alpha_1 \mathbf{x} = \alpha_1[m_1 + 2n_1, m_1, n_1] \\
= [\alpha_1(m_1 + 2n_1), \alpha_1 m_1, \alpha_1 n_1]
\]

Since \( \alpha_1(m_1 + 2n_1) \) and \( \alpha_1 m_1 \) and \( \alpha_1 n_1 \) are members of \( \mathbb{R} \), \( \alpha_1 \mathbf{x} \in \mathcal{V} \), and property M1 is satisfied.

\[
(\alpha_1 \beta_1) \mathbf{x} = (\alpha_1 \beta_1)[m_1 + 2n_1, m_1, n_1] \\
= [\alpha_1 \beta_1(m_1 + 2n_1), \alpha_1 \beta_1 m_1, \alpha_1 \beta_1 n_1] \\
= \alpha_1[\beta_1(m_1 + 2n_1), \beta_1 m_1, \beta_1 n_1] \\
= \alpha_1(\beta_1 \mathbf{x})
\]

So property M2 is satisfied.

\[
\alpha_1(\mathbf{x} + \mathbf{y}) = \alpha_1([m_1 + 2n_1, m_1, n_1] + [m_2 + 2n_2, m_2, n_2]) \\
= \alpha_1[m_1 + 2n_1 + m_2 + 2n_2, m_1 + m_2, n_1 + n_2] \\
= [\alpha_1(m_1 + 2n_1 + m_2 + 2n_2), \alpha_1(m_1 + m_2), \alpha_1(n_1 + n_2)] \\
= [\alpha_1(m_1 + 2n_1) + \alpha_1(m_2 + 2n_2), \alpha_1 m_1 + \alpha_1 m_2, \alpha_1 n_1 + \alpha_1 n_2] \\
= [\alpha_1(m_1 + 2n_1), \alpha_1 m_1, n_1] + [\alpha_1(m_2 + 2n_2), \alpha_1 m_2, \alpha_1 n_2] \\
= \alpha_1[m_1 + 2n_1, m_1, n_1] + \alpha_1[m_2 + 2n_2, m_2, n_2] \\
= \alpha_1 \mathbf{x} + \alpha_1 \mathbf{y}
\]

So property M3 is satisfied.

\[
(\alpha_1 + \beta_1) \mathbf{x} = (\alpha_1 + \beta_1)[m_1 + 2n_1, m_1, n_1] \\
= [(\alpha_1 + \beta_1)(m_1 + 2n_1), (\alpha_1 + \beta_1)m_1, (\alpha_1 + \beta_1)n_1] \\
= [\alpha_1(m_1 + 2n_1) + \beta_1(m_1 + 2n_1), \alpha_1 m_1 + \beta_1 m_1, \alpha_1 n_1 + \beta_1 n_1] \\
= [\alpha_1(m_1 + 2n_1), \alpha_1 m_1, \alpha_1 n_1] + [\beta_1(m_1 + 2n_1), \beta_1 m_1, \beta_1 n_1] \\
= \alpha_1[m_1 + 2n_1, m_1, n_1] + \beta_1[m_1 + 2n_1, m_1, n_1] \\
= \alpha_1 \mathbf{x} + \beta_1 \mathbf{x}
\]

So property M4 is satisfied.

\[
1 \mathbf{x} = 1[m_1 + 2n_1, m_1, n_1] = [1(m_1 + 2n_1), 1 \times m_1, 1 \times n_1] = [m_1 + 2n_1, m_1, n_1] = \mathbf{x}
\]

So property M5 is satisfied.

All ten properties are satisfied, so \( \mathcal{V} = \{[a, b, c] \mid m[1, 1, 0] + n[2, 0, 1] = [a, b, c] \} \) is a vector space.
Part (e)

\[ V = \{ [a, b, c] \mid c - a = 2b, \ a, c \in \mathbb{R} \} \]

Choose \( x = [a_1, \frac{1}{2}(c_1 - a_1), c_1] \in V \) and \( y = [a_2, \frac{1}{2}(c_2 - a_2), c_2] \in V \) and \( z = [a_3, \frac{1}{2}(c_3 - a_3), c_3] \in V \).

\[
x + y = \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] + \left[ a_2, \frac{1}{2}(c_2 - a_2), c_2 \right] = \left[ a_1 + a_2, \frac{1}{2}\{ (c_1 + c_2) - (a_1 + a_2) \}, c_1 + c_2 \right] = \left[ a_4, \frac{1}{2}(c_4 - a_4), c_4 \right]
\]

Because \( a_4 \in \mathbb{R} \) and \( c_4 \in \mathbb{R} \) and the second component is \( \frac{1}{2}(c_4 - a_4) \), \( x + y \in V \). Hence, property A1 is satisfied.

\[
(x + y) + z = \left( \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] + \left[ a_2, \frac{1}{2}(c_2 - a_2), c_2 \right] \right) + \left[ a_3, \frac{1}{2}(c_3 - a_3), c_3 \right] = \left[ a_1 + a_2 + a_3, \frac{1}{2}\{ (c_1 + c_2 + c_3) - (a_1 + a_2 + a_3) \}, c_1 + c_2 + c_3 \right] = \left[ a_1 + a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] + \left[ a_2 + a_3, \frac{1}{2}\{ (c_2 + c_3) - (a_2 + a_3) \}, c_2 + c_3 \right] = \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] + \left( \left[ a_2, \frac{1}{2}(c_2 - a_2), c_2 \right] + \left[ a_3, \frac{1}{2}(c_3 - a_3), c_3 \right] \right) = x + (y + z)
\]

Property A2 is satisfied.

\[
x + y = \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] + \left[ a_2, \frac{1}{2}(c_2 - a_2), c_2 \right] = \left[ a_1 + a_2, \frac{1}{2}\{ (c_1 + c_2) - (a_1 + a_2) \}, c_1 + c_2 \right] = \left[ a_2 + a_1, \frac{1}{2}(c_2 + c_1) - (a_2 + a_1), c_2 + c_1 \right] = \left[ a_2, \frac{1}{2}(c_2 - a_2), c_2 \right] + \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] = y + x
\]

Property A3 is satisfied. Setting \( a = 0 \) and \( c = 0 \), we find that \( 0 = [0, 0, 0] \in V \), so property A4 is satisfied. Since \( -a_1 \) and \( -c_1 \) are members of \( \mathbb{R} \), \( -x \in V \), and property A5 is satisfied.

Choose \( \alpha_1 \in \mathbb{R} \) and \( \beta_1 \in \mathbb{R} \).

\[
\alpha_1 x = \alpha_1 \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] = \left[ \alpha_1 a_1, \frac{1}{2}\alpha_1(c_1 - a_1), \alpha_1 c_1 \right]
\]
Since \( \alpha_1 a_1 \) and \( \alpha_1 c_1 \) and \( \frac{1}{2} \alpha_1 (c_1 - a_1) \) are members of \( \mathbb{R} \), \( \alpha_1 x \in V \), and property M1 is satisfied.

\[
(\alpha_1 \beta_1)x = (\alpha_1 \beta_1) \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] = \alpha_1 \left[ \alpha_1 \beta_1 a_1, \frac{1}{2} \alpha_1 \beta_1 (c_1 - a_1), \alpha_1 \beta_1 c_1 \right] = \alpha_1 \left[ \beta_1 a_1, \frac{1}{2} \beta_1 (c_1 - a_1), \beta_1 c_1 \right] = \alpha_1 (\beta_1 x)
\]

So property M2 is satisfied.

\[
\alpha_1(x + y) = \alpha_1 \left( \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] + \left[ a_2, \frac{1}{2}(c_2 - a_2), c_2 \right] \right) = \alpha_1 \left[ a_1 + a_2, \frac{1}{2} ((c_1 + c_2) - (a_1 + a_2)), c_1 + c_2 \right] = \alpha_1 (a_1 + a_2, \frac{1}{2} \alpha_1 ((c_1 + c_2) - (a_1 + a_2)), \alpha_1 (c_1 + c_2)) = \left[ \alpha_1 a_1 + \alpha_1 a_2, \frac{1}{2} \alpha_1 (c_1 - a_1) + \frac{1}{2} \alpha_1 (c_2 - a_2), \alpha_1 c_1 + \alpha_1 c_2 \right] = \alpha_1 \left[ a_1, \frac{1}{2} (c_1 - a_1), c_1 \right] + \alpha_1 \left[ a_2, \frac{1}{2} (c_2 - a_2), c_2 \right] = \alpha_1 x + \alpha_1 y
\]

So property M3 is satisfied.

\[
(\alpha_1 + \beta_1)x = (\alpha_1 + \beta_1) \left[ a_1, \frac{1}{2}(c_1 - a_1), c_1 \right] = \left[ (\alpha_1 + \beta_1) a_1, \frac{1}{2} (\alpha_1 + \beta_1) (c_1 - a_1), (\alpha_1 + \beta_1) c_1 \right] = \left[ \alpha_1 a_1 + \beta_1 a_1, \frac{1}{2} \alpha_1 (c_1 - a_1) + \frac{1}{2} \beta_1 (c_1 - a_1), \alpha_1 c_1 + \beta_1 c_1 \right] = \left[ \alpha_1 a_1, \frac{1}{2} \alpha_1 (c_1 - a_1), \alpha_1 c_1 \right] + [\beta_1 a_1, \frac{1}{2} \beta_1 (c_1 - a_1), \beta_1 c_1] = \alpha_1 \left[ a_1, \frac{1}{2} (c_1 - a_1), c_1 \right] + \beta_1 \left[ a_1, \frac{1}{2} (c_1 - a_1), c_1 \right] = \alpha_1 x + \beta_1 x
\]

So property M4 is satisfied.

\[
1x = 1[a_1, \frac{1}{2}(c_1 - a_1), c_1] = [1 \times a_1, 1 \times \frac{1}{2}(c_1 - a_1)], 1 \times c_1 = [a_1, \frac{1}{2} (c_1 - a_1), c_1] = x, \text{ so property M5 is satisfied. All ten properties are satisfied, so } V = \{[a, b, c] \mid c - a = 2b, a, c \in \mathbb{R} \} \text{ is a vector space.}
\]