Exercise 7

Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?

Solution

To determine whether a set of functions $S = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ is linearly independent or not, one must consider the Wronski matrix.

$$W(x) = \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f'_1(x) & f'_2(x) & \cdots & f'_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)}_1(x) & f^{(n-1)}_2(x) & \cdots & f^{(n-1)}_n(x) \end{bmatrix}$$

If there is a point $x = x_0$ such that $W(x_0)$ is nonsingular (i.e. the determinant is nonzero), then the set of functions is linearly independent.

The associated Wronski matrix for the set of functions in this exercise is

$$W(x) = \begin{bmatrix} 1 + x & 1 - x & 1 + x + x^2 \\ 1 & -1 & 1 + 2x \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det W(x) = \begin{vmatrix} 1 + x & 1 - x & 1 + x + x^2 \\ 1 & -1 & 1 + 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$= (1 + x) \begin{vmatrix} -1 & 1 + 2x \\ 0 & 2 \end{vmatrix} - (1 - x) \begin{vmatrix} 1 & 1 + 2x \\ 0 & 2 \end{vmatrix} + (1 + x + x^2) \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= (1 + x)(-2) - (1 - x)(2) + (1 + x + x^2)(0)$$

$$= -4$$

The determinant of $W(x)$ is nonzero for all values of $x$, so $W(x)$ is nonsingular. Therefore, the set of functions $\{1 + x, 1 - x, 1 + x + x^2\}$ is linearly independent.