Exercise 9

Show that the functions \((c_1 + c_2 \sin^2 x + c_3 \cos^2 x)\) form a vector space. Find a basis of it. What is its dimension?

Solution

This is a linear combination of the functions 1, \(\sin^2 x\), and \(\cos^2 x\). Since \(\sin^2 x + \cos^2 x = 1\), the basis is just \(\{\sin^2 x, \cos^2 x\}\). And because the basis has two components to it, the dimension is 2.

In order to show that the functions form a vector space, we must show that the following ten properties are satisfied:

(A1) \(x + y \in \mathcal{V}\) for all \(x, y \in \mathcal{V}\).

(A2) \((x + y) + z = x + (y + z)\) for every \(x, y, z \in \mathcal{V}\).

(A3) \(x + y = y + x\) for every \(x, y \in \mathcal{V}\).

(A4) There is an element \(0 \in \mathcal{V}\) such that \(x + 0 = x\) for every \(x \in \mathcal{V}\).

(A5) For each \(x \in \mathcal{V}\), there is an element \((-x) \in \mathcal{V}\) such that \(x + (-x) = 0\).

(M1) \(\alpha x \in \mathcal{V}\) for all \(\alpha \in \mathbb{R}\) and \(x \in \mathcal{V}\).

(M2) \((\alpha \beta)x = \alpha(\beta x)\) for all \(\alpha, \beta \in \mathbb{R}\) and every \(x \in \mathcal{V}\).

(M3) \(\alpha(x + y) = \alpha x + \alpha y\) for every \(\alpha \in \mathbb{R}\) and all \(x, y \in \mathcal{V}\).

(M4) \((\alpha + \beta)x = \alpha x + \beta x\) for all \(\alpha, \beta \in \mathbb{R}\) and every \(x \in \mathcal{V}\).

(M5) \(1x = x\) for every \(x \in \mathcal{V}\).

Suppose \(x \in \mathcal{V}\) and \(y \in \mathcal{V}\). Then

\[
x = a_1 + b_1 \sin^2 x + c_1 \cos^2 x
\]

\[
y = a_2 + b_2 \sin^2 x + c_2 \cos^2 x
\]

The sum is \(x + y\).

\[
x + y = a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x
\]

\[
= (a_1 + a_2) + (b_1 + b_2) \sin^2 x + (c_1 + c_2) \cos^2 x
\]

\[
= a + b \sin^2 x + c \cos^2 x
\]

Because \(a, b, c \in \mathbb{R}\), \(x + y\) is just another vector in \(\mathcal{V}\); that is, \(x + y \in \mathcal{V}\), and property A1 is satisfied. Suppose \(z \in \mathcal{V}\). Then \(z = a_3 + b_3 \sin^2 x + c_3 \cos^2 x\), and

\[
(x + y) + z = (a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x) + a_3 + b_3 \sin^2 x + c_3 \cos^2 x
\]

\[
= a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x + a_3 + b_3 \sin^2 x + c_3 \cos^2 x
\]

\[
= a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x + a_3 + b_3 \sin^2 x + c_3 \cos^2 x
\]

\[
= x + (y + z)
\]
So property A2 is satisfied.
\[
\mathbf{x} + \mathbf{y} = a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x \\
= a_2 + b_2 \sin^2 x + c_2 \cos^2 x + a_1 + b_1 \sin^2 x + c_1 \cos^2 x \\
= \mathbf{y} + \mathbf{x}
\]

So property A3 is satisfied. We can choose \( a_1 = b_1 = c_1 = 0 \) to get the zero vector, so property A4 is satisfied. We can choose \( a_2 = -a_1, b_2 = -b_1, \) and \( c_2 = -c_1 \) in \( \mathbf{y} \) to get \(-\mathbf{x}\).
\[
\mathbf{x} + (-\mathbf{x}) = a_1 + b_1 \sin^2 x + c_1 \cos^2 x + (-a_1 - b_1 \sin^2 x - c_1 \cos^2 x) \\
= a_1 + b_1 \sin^2 x + c_1 \cos^2 x - a_1 - b_1 \sin^2 x - c_1 \cos^2 x \\
= 0
\]

So property A5 is satisfied. Choose \( \alpha \in \mathbb{R} \). Then
\[
\alpha \mathbf{x} = \alpha (a_1 + b_1 \sin^2 x + c_1 \cos^2 x) \\
= \alpha a_1 + \alpha b_1 \sin^2 x + \alpha c_1 \cos^2 x \\
= a + b \sin^2 x + c \cos^2 x
\]

Thus, \( \alpha \mathbf{x} \) is just another vector in \( \mathcal{V} \). So property M1 is satisfied. Choose \( \beta \in \mathbb{R} \). Then
\[
(\alpha \beta) \mathbf{x} = (\alpha \beta) (a_1 + b_1 \sin^2 x + c_1 \cos^2 x) \\
= \alpha \beta a_1 + \alpha \beta b_1 \sin^2 x + \alpha \beta c_1 \cos^2 x \\
= \alpha (\beta a_1 + \beta b_1 \sin^2 x + \beta c_1 \cos^2 x) \\
= \alpha (\beta \mathbf{x})
\]

So property M2 is satisfied.
\[
\alpha (\mathbf{x} + \mathbf{y}) = \alpha (a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x) \\
= \alpha a_1 + \alpha b_1 \sin^2 x + \alpha c_1 \cos^2 x + \alpha a_2 + \alpha b_2 \sin^2 x + \alpha c_2 \cos^2 x \\
= \alpha (a_1 + b_1 \sin^2 x + c_1 \cos^2 x) + \alpha (a_2 + b_2 \sin^2 x + c_2 \cos^2 x) \\
= \alpha \mathbf{x} + \alpha \mathbf{y}
\]

So property M3 is satisfied.
\[
(\alpha + \beta) \mathbf{x} = (\alpha + \beta) (a_1 + b_1 \sin^2 x + c_1 \cos^2 x) \\
= (\alpha + \beta) a_1 + (\alpha + \beta) b_1 \sin^2 x + (\alpha + \beta) c_1 \cos^2 x \\
= \alpha a_1 + \beta a_1 + \alpha b_1 \sin^2 x + \beta b_1 \sin^2 x + \alpha c_1 \cos^2 x + \beta c_1 \cos^2 x \\
= \alpha a_1 + \alpha b_1 \sin^2 x + \alpha c_1 \cos^2 x + \beta a_1 + \beta b_1 \sin^2 x + \beta c_1 \cos^2 x \\
= \alpha (a_1 + b_1 \sin^2 x + c_1 \cos^2 x) + \beta (a_2 + b_2 \sin^2 x + c_2 \cos^2 x) \\
= \alpha \mathbf{x} + \beta \mathbf{y}
\]

So property M4 is satisfied.
\[
1 \mathbf{x} = 1 (a_1 + b_1 \sin^2 x + c_1 \cos^2 x) \\
= 1 \cdot a_1 + 1 \cdot b_1 \sin^2 x + 1 \cdot c_1 \cos^2 x \\
= a_1 + b_1 \sin^2 x + c_1 \cos^2 x \\
= \mathbf{x}
\]

So property M5 is satisfied. All ten properties are satisfied, and so the functions 
\((c_1 + c_2 \sin^2 x + c_3 \cos^2 x)\) form a vector space.

www.stemjock.com