Exercise 10

Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

Solution

The Geometric Method: Characteristic Curves

On the paths defined by

$$\frac{dy}{dx} = 1, \quad y(\xi, 0) = \xi, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} + u = e^{x+2y}. \tag{2}$$

That is, $u = u(x,y)$ is constant on the characteristics defined by (1). (2) is a first-order linear ODE that can be solved with an integrating factor. Multiply both sides by

$$I = e^{\int 1\,dx} = e^x.$$

We find that

$$\begin{aligned} e^x \frac{du}{dx} + e^x u &= e^x e^{x+2y} \\ \frac{d}{dx}(e^x u) &= e^{2x+2y} \end{aligned}$$

Now integrate both sides.

$$e^x u = \int e^{2(x+y)}\,dx + f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Before we go any further, it is critical to note that y is a function of ξ and x, so we have to solve (1) to proceed. Integrating (1) gives

$$y = x + \xi. \tag{3}$$

Substituting this into the integral yields

$$\begin{aligned} e^x u &= \int e^{2(2x+\xi)}\,dx + f(\xi) \\ e^x u &= \frac{1}{4} e^{2(2x+\xi)} + f(\xi) \\ u(x,\xi) &= \frac{1}{4} e^{3x+2\xi} + e^{-x} f(\xi), \end{aligned}$$

Solving (3) for ξ gives

$$\xi = y - x.$$

And so

$$u(x,y) = \frac{1}{4} e^{3x+2(y-x)} + e^{-x} f(y-x).$$

Simplifying this gives us

$$u(x,y) = \frac{1}{4} e^{x+2y} + e^{-x} f(y-x).$$

We're told that $u(x,0) = 0$, though, so we can determine this unknown function, f.

$$u(x,0) = \frac{1}{4}e^{x} + e^{-x}f(-x) = 0$$

Solving for f gives

$$f(-x) = -\frac{1}{4}e^{2x}.$$

This implies that $f(w) = -\frac{1}{4}e^{-2w}$, where w is any expression. Thus,

$$\begin{aligned} u(x,y) &= \frac{1}{4}e^{x+2y} + e^{-x}\left[-\frac{1}{4}e^{-2(y-x)}\right] \\ u(x,y) &= \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right). \end{aligned} \tag{4}$$

Another form for $u(x,y)$ can be obtained by factoring out e^{x}.

$$u(x,y) = \frac{e^{x}}{2}\left(\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y}\right)$$

And so

$$u(x,y) = \frac{1}{2}e^{x}\sinh 2y.$$

We can check that (4) is the solution to the PDE.

$$\begin{aligned} u_x &= \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right) \\ u_y &= \frac{1}{4}\left(2e^{x+2y} + 2e^{x-2y}\right) \end{aligned}$$

Hence,

$$\begin{aligned} u_x + u_y + u &= \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right) + \frac{1}{4}\left(2e^{x+2y} + 2e^{x-2y}\right) + \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right) \\ u_x + u_y + u &= \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right)e^{x+2y} + \left(-\frac{1}{4} + \frac{1}{2} - \frac{1}{4}\right)e^{x-2y} \\ u_x + u_y + u &= e^{x+2y}, \end{aligned}$$

which means that (4) is the solution to the PDE. Shown below in Figure 1 are the characteristic curves in the xy-plane. Since the data curve ($y = 0$) intersects each of the characteristics exactly once, the solution we obtained for the PDE is valid everywhere in the xy-plane, that is, for all x and y. $u(x,y)$ is shown in Figure 2.

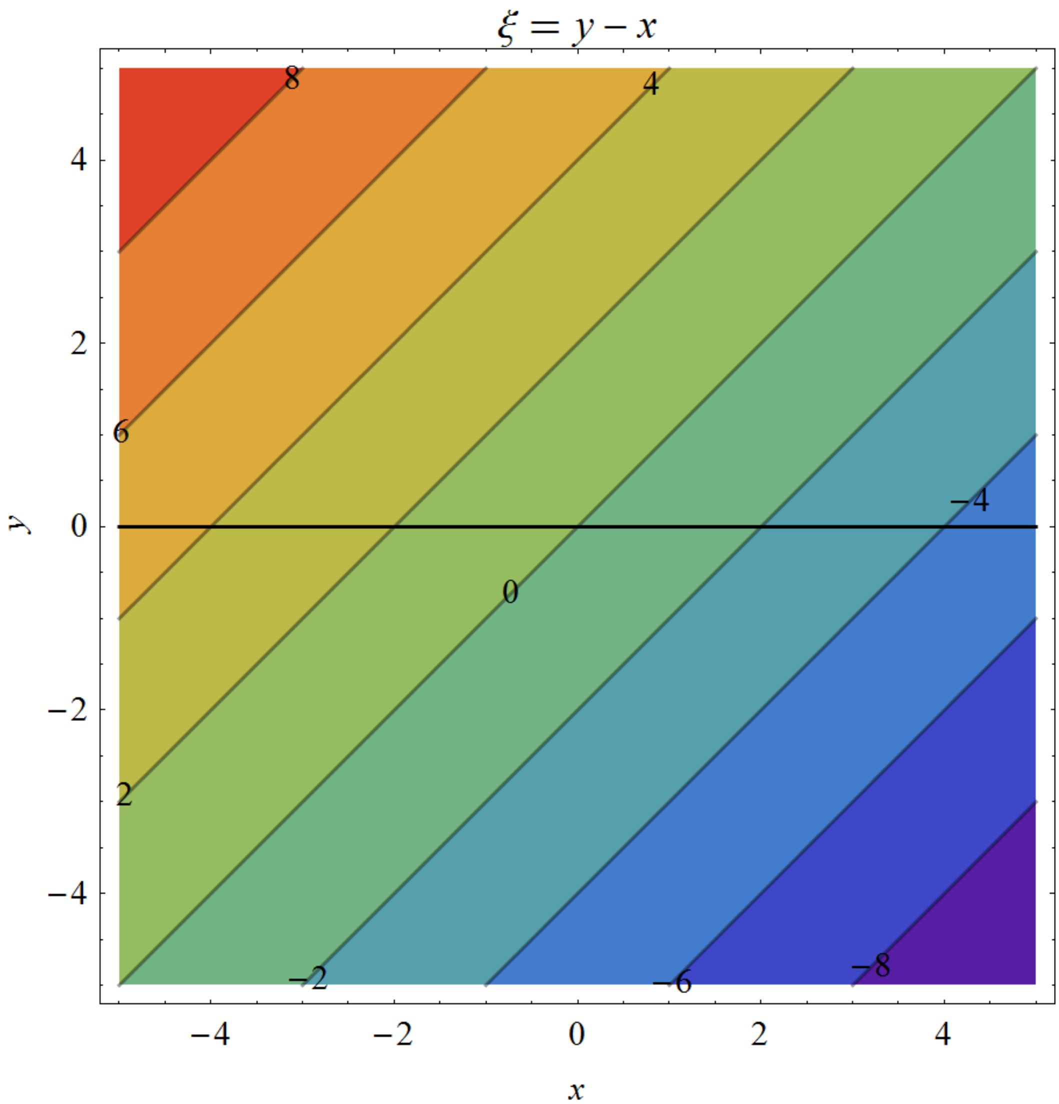

Figure 1: Plot of the characteristic curves in the xy-plane along with the data curve for $-5 < x < 5$ and $-5 < y < 5$.

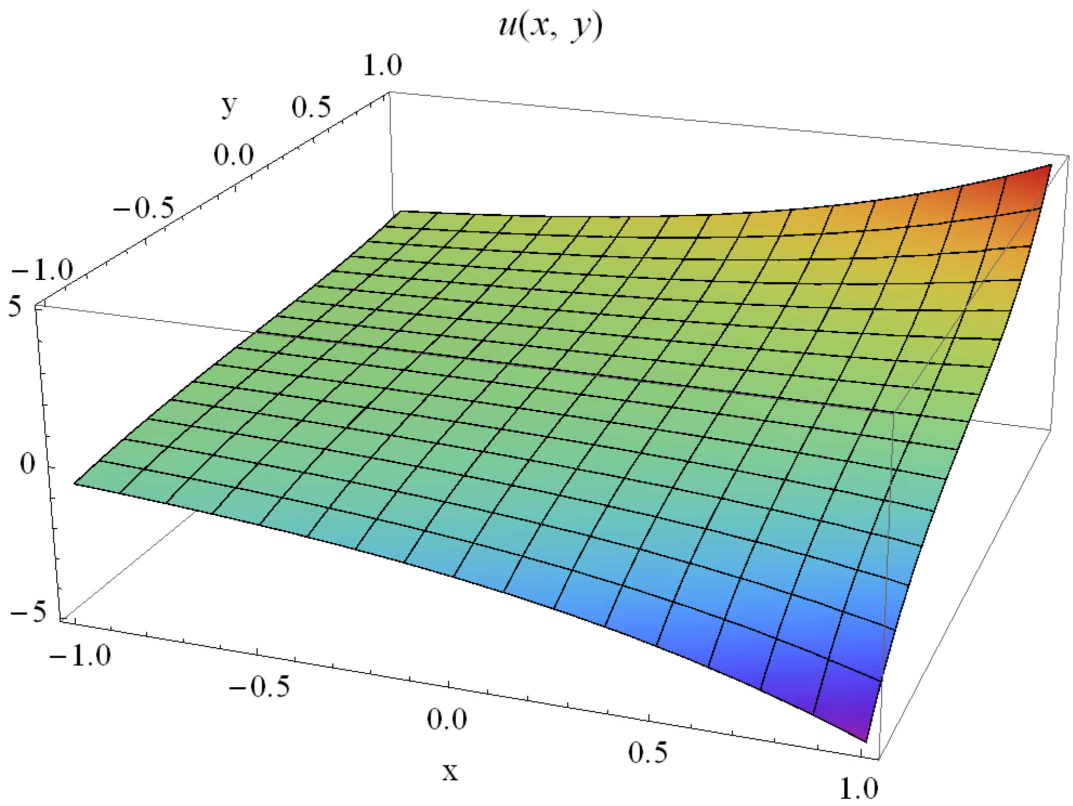

Figure 2: Plot of $u(x, y)$ for $-1 < x < 1$ and $-1 < y < 1$.

The Coordinate Method: Change of Variables

To solve this PDE with the coordinate method, start by making the change of variables,

$$\begin{aligned} x' &= x + y \\ y' &= x - y. \end{aligned}$$

Solving for the old variables in terms of the new ones gives us

$$\begin{aligned} x &= \frac{1}{2}(x' + y') \\ y &= \frac{1}{2}(x' - y'). \end{aligned}$$

To find what u_x and u_y are in terms of these new variables, it's necessary to use the chain rule.

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'} + u_{y'} \\ u_y &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = u_{x'} - u_{y'} \end{aligned}$$

Now we substitute these expressions into the PDE. The transformed equation is

$$(u_{x'} + u_{y'}) + (u_{x'} - u_{y'}) + u = e^{\frac{1}{2}(x'+y')+2\left[\frac{1}{2}(x'-y')\right]}.$$

Simplifying this gives

$$2u_{x'} + u = e^{\frac{1}{2}(3x'-y')}$$
$$u_{x'} + \frac{1}{2}u = \frac{1}{2}e^{\frac{1}{2}(3x'-y')}.$$

This is a linear inhomogeneous equation for u, so we have to multiply both sides by the integrating factor,

$$I(x') = e^{\int^{x'} \frac{1}{2}\,ds} = e^{\frac{1}{2}x'}.$$
$$e^{\frac{1}{2}x'}u_{x'} + \frac{1}{2}e^{\frac{1}{2}x'}u = \frac{1}{2}e^{\frac{1}{2}(4x'-y')}$$

Now the left side can be written as the x'-derivative of Iu.

$$\frac{\partial}{\partial x'}\left(e^{\frac{1}{2}x'}u\right) = \frac{1}{2}e^{\frac{1}{2}(4x'-y')}$$

Partially integrate both sides with respect to x'.

$$e^{\frac{1}{2}x'}u = \int^{x'} \frac{1}{2}e^{\frac{1}{2}(4s-y')}\,ds + g(y'),$$

where g is an arbitrary function of y'.

$$e^{\frac{1}{2}x'}u = \frac{1}{4}e^{\frac{1}{2}(4x'-y')} + g(y')$$
$$u(x',y') = \frac{1}{4}e^{\frac{1}{2}(3x'-y')} + e^{-\frac{1}{2}x'}g(y')$$

Now we return to the original variables, x and y.

$$u(x,y) = \frac{1}{4}e^{\frac{1}{2}[3(x+y)-(x-y)]} + e^{-\frac{1}{2}(x+y)}g(x-y)$$

Simplifying this result gives

$$u(x,y) = \frac{1}{4}e^{x+2y} + e^{-\frac{1}{2}(x+y)}g(x-y).$$

We're told that $u(x,0) = 0$, though, so we can determine this unknown function, g.

$$u(x,0) = \frac{1}{4}e^{x} + e^{-\frac{1}{2}x}g(x) = 0$$
$$e^{-\frac{1}{2}x}g(x) = -\frac{1}{4}e^{x}$$
$$g(x) = -\frac{1}{4}e^{\frac{3}{2}x}$$

This implies that

$$g(w) = -\frac{1}{4}e^{\frac{3}{2}w},$$

where w is any expression. Thus,

$$u(x,y) = \frac{1}{4}e^{x+2y} + e^{-\frac{1}{2}(x+y)}\left[-\frac{1}{4}e^{\frac{3}{2}(x-y)}\right]$$
$$u(x,y) = \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right),$$

which is the same answer we obtained using the method of characteristics.