Exercise 8

Solve $au_x + bu_y + cu = 0$.

Solution

The Geometric Method: Characteristic Curves

Start by rewriting the PDE as

$$u_x + \frac{b}{a}u_y = -\frac{c}{a}u$$

and then apply the method of characteristics to solve for u. On the paths defined by

$$\frac{dy}{dx} = \frac{b}{a}, \quad y(\xi, 0) = \xi, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = -\frac{c}{a}u. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Solving (2) by separation of variables, we find that

$$\begin{aligned}
\frac{du}{u} &= -\frac{c}{a}\,dx \\
\ln|u| &= -\frac{c}{a}x + C(\xi) \\
|u| &= e^{-\frac{c}{a}x}e^{C(\xi)} \\
u(x, \xi) &= \pm e^{C(\xi)}e^{-\frac{c}{a}x} \\
u(x, \xi) &= f(\xi)e^{-\frac{c}{a}x},
\end{aligned}$$

where f is an arbitrary function of the characteristic coordinate, ξ. Integrating (1) gives

$$y = \frac{b}{a}x + \xi.$$

Solving for ξ gives

$$\xi = y - \frac{b}{a}x.$$

Therefore,

$$u(x, y) = f\left(y - \frac{b}{a}x\right)e^{-\frac{c}{a}x}.$$

We can check that this is the solution to the PDE.

$$\begin{aligned}
u_x &= -\frac{b}{a}e^{-\frac{c}{a}x}f' - \frac{c}{a}e^{-\frac{c}{a}x}f \\
u_y &= e^{-\frac{c}{a}x}f'
\end{aligned}$$

$$\begin{aligned}
au_x + bu_y + cu &= \cancel{-be^{-\frac{c}{a}x}f'} - \cancel{ce^{-\frac{c}{a}x}f} + \cancel{be^{-\frac{c}{a}x}f'} + \cancel{ce^{-\frac{c}{a}x}f} \\
&= 0,
\end{aligned}$$

The Coordinate Method: Change of Variables

To solve this PDE with the coordinate method, start by making the change of variables,

$$x' = ax + by$$
$$y' = bx - ay.$$

Solving for the old variables in terms of the new ones gives us

$$x = \frac{ax' + by'}{a^2 + b^2}$$
$$y = \frac{bx' - ay'}{a^2 + b^2}.$$

To find what u_x and u_y are in terms of these new variables, it's necessary to use the chain rule.

$$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$
$$u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'}$$

Now we substitute these expressions into the PDE. The transformed equation is

$$a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) + cu = 0.$$

Simplifying this gives

$$(a^2 + b^2)u_{x'} + cu = 0.$$

Now isolate $u_{x'}$.

$$u_{x'} = -\frac{c}{a^2 + b^2}u$$

Solve for u by separating variables and partially integrating both sides with respect to x'.

$$\frac{\partial u}{\partial x'} = -\frac{c}{a^2 + b^2}u$$
$$\frac{\partial u}{u} = -\frac{c}{a^2 + b^2}\,\partial x'$$
$$\ln|u| = -\frac{c}{a^2 + b^2}x' + g(y')$$
$$|u| = e^{-\frac{c}{a^2+b^2}x' + g(y')}$$
$$u(x', y') = \pm e^{g(y')}e^{-\frac{c}{a^2+b^2}x'}$$
$$u(x', y') = h(y')e^{-\frac{c}{a^2+b^2}x'},$$

where h is an arbitrary function of y'. Now return back to the original variables, x and y.

$$u(x, y) = h(bx - ay)e^{-\frac{c}{a^2+b^2}(ax+by)}$$

We can check that this is the solution.

$$u_x = be^{-\frac{c}{a^2+b^2}(ax+by)}h' - \frac{ac}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h$$
$$u_y = -ae^{-\frac{c}{a^2+b^2}(ax+by)}h' - \frac{bc}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h$$

And so

$$\begin{aligned}
au_x + bu_y + cu &= \cancel{abe^{-\frac{c}{a^2+b^2}(ax+by)}}h' - \frac{a^2c}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h \\
&\quad - \cancel{abe^{-\frac{c}{a^2+b^2}(ax+by)}}h' - \frac{b^2c}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h + ce^{-\frac{c}{a^2+b^2}(ax+by)}h \\
au_x + bu_y + cu &= -c\frac{\cancel{a^2+b^2}}{\cancel{a^2+b^2}}e^{-\frac{c}{a^2+b^2}(ax+by)}h + ce^{-\frac{c}{a^2+b^2}(ax+by)}h \\
au_x + bu_y + cu &= 0.
\end{aligned}$$