Exercise 1

Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

Solution

**The Geometric Method: Characteristic Curves**

Start by rewriting the PDE as

$$u_t + \frac{3}{2} u_x = 0$$

and then apply the method of characteristics to solve for $u$. On the paths defined by

$$\frac{dx}{dt} = \frac{3}{2}, \quad x(\xi,0) = \xi,$$  \hspace{1cm} (1)

the PDE reduces to an ODE,

$$\frac{du}{dt} = 0.$$  \hspace{1cm} (2)

That is, $u = u(x,t)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(\xi,t) = f(\xi),$$

where $f$ is an arbitrary function of the characteristic coordinate, $\xi$. Integrating (1), we see that

$$x = \frac{3}{2} t + \xi.$$  

Solving for $\xi$ gives

$$\xi = x - \frac{3}{2} t.$$  

Therefore,

$$u(x,t) = f \left( x - \frac{3}{2} t \right).$$

We can check that this is the solution to the PDE.

$$u_x = f'$$

$$u_t = -\frac{3}{2} f'.$$

$2u_t + 3u_x = 0$, so this is the solution to the PDE. We’re told that $u(x,0) = \sin x$, though, so we can determine this unknown function, $f$.

$$u(x,0) = f(x) = \sin x$$

This implies that $f(w) = \sin w$, where $w$ is any expression. Thus,

$$u(x,t) = \sin \left( x - \frac{3}{2} t \right).$$

The function is shown below in Figure 1. Shown below that in Figure 2 are the characteristic curves in the $tx$-plane for various values of $\xi$ along with the line $t = 0$ (where the auxiliary condition is defined). Note that because $t = 0$ intersects each of the characteristics exactly once, the solution we obtained for $u(x,t)$ is valid everywhere in the $tx$-plane, that is, for all $x$ and $t.$

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Figure 1: Plot of $u(x, t)$ for $-5 < t < 5$ and $-5 < x < 5$. 
The Coordinate Method: Change of Variables

To solve this PDE with the coordinate method, start by making the change of variables,

\[ t' = 2t + 3x \]
\[ x' = 3t - 2x. \]

Solving for the old variables in terms of the new ones gives us

\[ t = \frac{1}{13}(2t' + 3x') \]
\[ x = \frac{1}{13}(3t' - 2x'). \]
To find what $u_t$ and $u_x$ are in terms of these new variables, it's necessary to use the chain rule.

$$u_t = \frac{\partial u}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial t} = 2u_t' + 3u_{x'}$$

$$u_x = \frac{\partial u}{\partial t'} \frac{\partial t'}{\partial x} + \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} = 3u_t' - 2u_{x'}$$

Now we substitute these expressions into the PDE. The transformed equation is

$$2(2u_t' + 3u_{x'}) + 3(3u_t' - 2u_{x'}) = 0.$$  

Simplifying this gives

$$13u_t' = 0$$

$$u_t' = 0.$$  

Solve for $u$ by partially integrating both sides with respect to $t'$.

$$u(x', t') = g(x'),$$

where $g$ is an arbitrary function of $x'$. Now we return to the original variables, $x$ and $t$.

$$u(x, t) = g(3t - 2x)$$

We can check that this is the solution.

$$u_t = 3g'$$

$$u_x = -2g'$$

$$2u_t + 3u_x = 0,$$  

which means this is the solution to the PDE. Now plug in the initial condition, $u(x, 0) = \sin x$ to determine $g$.

$$u(x, 0) = g(-2x) = \sin x$$

This implies that

$$g(w) = \sin \left( -\frac{w}{2} \right),$$

where $w$ is any expression. Therefore,

$$u(x, t) = \sin \left( -\frac{3t - 2x}{2} \right)$$

$$u(x, t) = \sin \left( x - \frac{3}{2}t \right),$$

and we get the same answer as with the method of characteristics.