Exercise 8

Solve $au_x + bu_y + cu = 0$.

Solution

The Geometric Method: Characteristic Curves

Start by rewriting the PDE as
\[
u_x + \frac{b}{a} u_y = -\frac{c}{a} u
\]
and then apply the method of characteristics to solve for $u$. On the paths defined by
\[
\frac{dy}{dx} = \frac{b}{a}, \quad y(\xi, 0) = \xi,
\]
the PDE reduces to an ODE,
\[
\frac{du}{dx} = -\frac{c}{a} u. \tag{2}
\]
That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Solving (2) by separation of variables, we find that
\[
\frac{du}{u} = -\frac{c}{a} dx
\]
\[
\ln |u| = -\frac{c}{a} x + C(\xi)
\]
\[
|u| = e^{-\frac{c}{a} x} e^{C(\xi)}
\]
\[
u(x, \xi) = \pm e^{C(\xi)} e^{-\frac{c}{a} x}
\]
\[
u(x, \xi) = f(\xi) e^{-\frac{c}{a} x},
\]
where $f$ is an arbitrary function of the characteristic coordinate, $\xi$. Integrating (1) gives
\[
y = \frac{b}{a} x + \xi
\]
Solving for $\xi$ gives
\[
\xi = y - \frac{b}{a} x.
\]
Therefore,
\[
u(x, y) = f \left( y - \frac{b}{a} x \right) e^{-\frac{c}{a} x}.
\]
We can check that this is the solution to the PDE.

\[
u_x = -\frac{b}{a} e^{-\frac{c}{a} x} f' - \frac{c}{a} e^{-\frac{c}{a} x} f
\]
\[
u_y = e^{-\frac{c}{a} x} f'
\]
\[
au_x + bu_y + cu = -be^{-\frac{c}{a} x} f' - ce^{-\frac{c}{a} x} f + be^{-\frac{c}{a} x} f' + ce^{-\frac{c}{a} x} f
\]
\[
= 0,
\]
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The Coordinate Method: Change of Variables

To solve this PDE with the coordinate method, start by making the change of variables,

\[ x' = ax + by \]
\[ y' = bx - ay. \]

Solving for the old variables in terms of the new ones gives us

\[ x = \frac{ax' + by'}{a^2 + b^2} \]
\[ y = \frac{bx' - ay'}{a^2 + b^2}. \]

To find what \( u_x \) and \( u_y \) are in terms of these new variables, it’s necessary to use the chain rule.

\[ u_x = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \]
\[ u_y = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \]

Now we substitute these expressions into the PDE. The transformed equation is

\[ a(a_{x'} + bu_{y'}) + b(b_{x'} - au_{y'}) + cu = 0. \]

Simplifying this gives

\[ (a^2 + b^2)u_{x'} + cu = 0. \]

Now isolate \( u_{x'} \).

\[ u_{x'} = -\frac{c}{a^2 + b^2}u \]

Solve for \( u \) by separating variables and partially integrating both sides with respect to \( x' \).

\[ \frac{\partial u}{\partial x'} = -\frac{c}{a^2 + b^2}u \]
\[ \frac{\partial u}{u} = -\frac{c}{a^2 + b^2} \partial x' \]
\[ \ln|u| = -\frac{c}{a^2 + b^2}x' + g(y') \]
\[ |u| = e^{-\frac{c}{a^2 + b^2}x'}e^{g(y')} \]
\[ u(x', y') = \pm e^{g(y')}e^{-\frac{c}{a^2 + b^2}x'} \]
\[ u(x', y') = h(y')e^{-\frac{c}{a^2 + b^2}x'}, \]

where \( h \) is an arbitrary function of \( y' \). Now return back to the original variables, \( x \) and \( y \).

\[ u(x, y) = h(bx - ay)e^{-\frac{c}{a^2 + b^2}(ax + by)} \]
We can check that this is the solution.

\[ u_x = be^{-\frac{c}{a^2+b^2}(ax+by)}h' - \frac{ac}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h \]

\[ u_y = -ae^{-\frac{c}{a^2+b^2}(ax+by)}h' - \frac{bc}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h \]

And so

\[ au_x + bu_y + cu = abe^{-\frac{c}{a^2+b^2}(ax+by)}h' - \frac{a^2c}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h \]
\[ - abe^{-\frac{c}{a^2+b^2}(ax+by)}h' = \frac{b^2c}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h + ce^{-\frac{c}{a^2+b^2}(ax+by)}h \]

\[ au_x + bu_y + cu = -\frac{a^2+b^2}{a^2+b^2}e^{-\frac{c}{a^2+b^2}(ax+by)}h + ce^{-\frac{c}{a^2+b^2}(ax+by)}h \]

\[ au_x + bu_y + cu = 0. \]