Exercise 9

Solve the equation $u_x + u_y = 1$.

Solution

The Geometric Method: Characteristic Curves

On the paths defined by
\[
\frac{dy}{dx} = 1, \quad y(\xi, 0) = \xi, \tag{1}
\]
the PDE reduces to an ODE,
\[
\frac{du}{dx} = 1. \tag{2}
\]
That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that
\[
u(x, \xi) = x + f(\xi).
\]
where $f$ is an arbitrary function of the characteristic coordinate, $\xi$. Integrating (1) gives
\[
y = x + \xi
\]
Solving for $\xi$ gives
\[
\xi = y - x.
\]
Therefore,
\[
u(x, y) = x + f(y - x).
\]
We can check that this is the solution.

\[
u_x = 1 - f'
\]
\[
u_y = f'
\]

$u_x + u_y = 1$, which means this is the solution to the PDE.
Figure 1: Plot of the characteristic curves in the $xy$-plane for $-5 < x < 5$ and $-5 < y < 5$.

The Coordinate Method: Change of Variables

To solve this PDE with the coordinate method, start by making the change of variables,

$$x' = x + y$$
$$y' = x - y.$$

Solving for the old variables in terms of the new ones gives us

$$x = \frac{1}{2}(x' + y')$$
$$y = \frac{1}{2}(x' - y').$$

To find what $u_x$ and $u_y$ are in terms of these new variables, it’s necessary to use the chain rule.

$$u_x = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x} = u_{x'} + u_{y'}$$

$$u_y = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y} = u_{x'} - u_{y'}$$

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Now we substitute these expressions into the PDE. The transformed equation is

\[
(u_{x'} + u_{y'}) + (u_{x'} - u_{y'}) = 1.
\]

Simplifying this gives

\[
2u_{x'} = 1 \\
u_{x'} = \frac{1}{2}.
\]

Solve for \( u \) by partially integrating both sides with respect to \( x' \).

\[
u(x', y') = \frac{1}{2}x' + g(y'),
\]

where \( g \) is an arbitrary function of \( y' \). Now we return to the original variables, \( x \) and \( y \).

\[
u(x, y) = \frac{1}{2}(x + y) + g(x - y)
\]

We can check that this is the solution.

\[
u_x = \frac{1}{2} + g' \\
u_y = \frac{1}{2} - g'
\]

\( u_x + u_y = 1 \), which means this is the solution to the PDE.