Exercise 11

Solve $au_x + bu_y = f(x, y)$, where $f(x, y)$ is a given function. If $a \neq 0$, write the solution in the form

$$u(x, y) = (a^2 + b^2)^{-1/2} \int_L f\,ds + g(bx - ay),$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x, y), and the integral is a line integral. (*Hint:* Use the coordinate method.)

Solution

Following the hint, we will use the coordinate method to prove this result. Start by making the change of variables,

$$\begin{aligned} x' &= ax + by \\ y' &= bx - ay. \end{aligned}$$

Solving for the old variables in terms of the new ones gives us

$$\begin{aligned} x &= \frac{ax' + by'}{a^2 + b^2} \\ y &= \frac{bx' - ay'}{a^2 + b^2}. \end{aligned}$$

To find what u_x and u_y are in terms of these new variables, it's necessary to use the chain rule.

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{aligned}$$

Now we substitute these expressions into the PDE. The transformed equation is

$$a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = f\left(\frac{ax' + by'}{a^2 + b^2}, \frac{bx' - ay'}{a^2 + b^2}\right).$$

Simplifying this gives

$$(a^2 + b^2)u_{x'} = f\left(\frac{ax' + by'}{a^2 + b^2}, \frac{bx' - ay'}{a^2 + b^2}\right).$$

Dividing both sides by $a^2 + b^2$ isolates $u_{x'}$.

$$u_{x'} = \frac{1}{a^2 + b^2} f\left(\frac{ax' + by'}{a^2 + b^2}, \frac{bx' - ay'}{a^2 + b^2}\right)$$

To solve for u, integrate both sides partially with respect to x'.

$$u(x', y') = \int^{x'} \frac{1}{a^2 + b^2} f\left(\frac{ar + by'}{a^2 + b^2}, \frac{br - ay'}{a^2 + b^2}\right) dr + g(y'),$$

where g is an arbitrary function of y'. Now we return to the original variables, x and y.

$$u(x, y) = \frac{1}{a^2 + b^2} \int^{ax+by} f\left[\frac{ar + b(bx - ay)}{a^2 + b^2}, \frac{br - a(bx - ay)}{a^2 + b^2}\right] dr + g(bx - ay)$$

Simplifying this result gives us

$$u(x,y) = \frac{1}{a^2+b^2}\int^{ax+by} f\left[\frac{b^2x+a(r-by)}{a^2+b^2}, \frac{a^2y+b(r-ax)}{a^2+b^2}\right] dr + g(bx-ay).$$

To write this in the form given in the problem statement, we need to have a factor of $(a^2+b^2)^{-1/2}$ in front. This implies that we need to make a substitution in the integral such that

$$\frac{dr}{\sqrt{a^2+b^2}} = ds,$$

which means s has to be

$$\frac{r}{\sqrt{a^2+b^2}} = s \quad \to \quad r = \sqrt{a^2+b^2}s.$$

This results in

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}}\int^{\frac{ax+by}{\sqrt{a^2+b^2}}} f\left[\frac{b^2x+a(\sqrt{a^2+b^2}s-by)}{a^2+b^2}, \frac{a^2y+b(\sqrt{a^2+b^2}s-ax)}{a^2+b^2}\right] ds + g(bx-ay), \quad (1)$$

which we can write compactly as

$$u(x,y) = (a^2+b^2)^{-1/2}\int_L f\, ds + g(bx-ay).$$

Note that the upper limit of integration can be thought of as the dot product between the unit vector pointing in the direction of the characteristics and the position vector of (x,y). That is, it is the component of the position vector that lies in the direction of the characteristic curves.

$$\frac{ax+by}{\sqrt{a^2+b^2}} = \left\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right\rangle \cdot \langle x, y\rangle$$

Is (1) the solution to the PDE? We will check to see that it does satisfy the equation. The easiest way to do this is to notice that $(a\partial_x + b\partial_y)$ of any function of $bx-ay$ is 0. If we apply this operator to both sides of (1), we get

$$(a\partial_x + b\partial_y)u(x,y) = \frac{1}{\sqrt{a^2+b^2}}\times (a\partial_x + b\partial_y)\int^{\frac{ax+by}{\sqrt{a^2+b^2}}} f\left[\frac{b^2x+a(\sqrt{a^2+b^2}s-by)}{a^2+b^2}, \frac{a^2y+b(\sqrt{a^2+b^2}s-ax)}{a^2+b^2}\right] ds + (a\partial_x + b\partial_y)g(bx-ay).$$

Since both arguments of f are functions of $bx-ay$, applying the operator to the integral just gives us the function inside with s evaluated at the upper limit of integration. But don't forget that this has to be multiplied by the operator applied to the upper limit of integration (the chain rule). $(a\partial_x + b\partial_y)g(bx-ay)$ is just $abg' - abg' = 0$. Therefore,

$$au_x + bu_y = \frac{1}{\sqrt{a^2+b^2}}\times f\left[\frac{b^2x+a(\sqrt{a^2+b^2}s-by)}{a^2+b^2}, \frac{a^2y+b(\sqrt{a^2+b^2}s-ax)}{a^2+b^2}\right]\Bigg|_{s=\frac{ax+by}{\sqrt{a^2+b^2}}} \left[(a\partial_x + b\partial_y)\frac{ax+by}{\sqrt{a^2+b^2}}\right]$$

$$au_x + bu_y = \frac{1}{\sqrt{a^2+b^2}} f\left[\frac{b^2x + a(ax)}{a^2+b^2}, \frac{a^2y + b(by)}{a^2+b^2}\right]\left[a\left(\frac{a}{\sqrt{a^2+b^2}}\right) + b\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right]$$

$$au_x + bu_y = \frac{1}{\sqrt{a^2+b^2}} f(x,y)\left(\frac{a^2+b^2}{\sqrt{a^2+b^2}}\right)$$

$$au_x + bu_y = f(x,y).$$

Hence, (1) is indeed the solution to the PDE, and it can be represented compactly as

$$u(x,y) = (a^2+b^2)^{-1/2}\int_L f\,ds + g(bx - ay).$$

Before we call it a night, let's have a little fun with this formula. Let's use it to solve Exercise 9, $u_x + u_y = 1$. Here $f(x,y) = 1$ and $a = b = 1$. If we plug these values into (1), we get

$$u(x,y) = \frac{1}{\sqrt{2}}\int^{\frac{x+y}{\sqrt{2}}} ds + g(x-y).$$

And so

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{2}}\frac{x+y}{\sqrt{2}} + g(x-y) \\ &= \frac{x}{2} + \frac{y}{2} + g(x-y). \end{aligned}$$

Although this form is different from the one derived in Exercise 9, it does satisfy the PDE.

$$u_x = \frac{1}{2} + g'$$

$$u_y = \frac{1}{2} - g'$$

$u_x + u_y = 1$, so this solution we obtained using formula (1) is valid. As another example, let's find the solution of $u_x + u_y = xy$. Here $f(x,y) = xy$ and $a = b = 1$. So formula (1) gives us

$$u(x,y) = \frac{1}{\sqrt{2}}\int^{\frac{x+y}{\sqrt{2}}}\left[\frac{x + (\sqrt{2}s - y)}{2}\right]\left[\frac{y + (\sqrt{2}s - x)}{2}\right] ds + g(x-y).$$

Performing the integration,

$$u(x,y) = \frac{1}{\sqrt{2}}\left\{\frac{s\left[2s^2 - 3(x-y)^2\right]}{12\sqrt{2}}\right\}\Bigg|_{s=\frac{x+y}{\sqrt{2}}} + g(x-y).$$

Plugging in $(x+y)/\sqrt{2}$ for s,

$$u(x,y) = -\frac{1}{12}(x+y)(x^2 - 4xy + y^2) + g(x-y).$$

And this is the solution to the example PDE. We can check it very quickly.

$$u_x = \frac{1}{4}(-x^2 + 2xy + y^2) + g'$$

$$u_y = \frac{1}{4}(x^2 + 2xy - y^2) - g'$$

Thus, $u_x + u_y = xy$. In conclusion, formula (1) works.