Exercise 13

Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

Solution

The Coordinate Method: Change of Variables

Because we're using the coordinate method, we have to start by making the change of variables,

$$\begin{aligned} x' &= ax + by = x + 2y \\ y' &= bx - ay = 2x - y. \end{aligned}$$

The original variables in terms of these new variables are

$$\begin{aligned} x &= \frac{1}{5}(x' + 2y') \\ y &= \frac{1}{5}(2x' - y'). \end{aligned}$$

To find what u_x and u_y are in terms of these new variables, it's necessary to use the chain rule.

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'} + 2u_{y'} \\ u_y &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 2u_{x'} - u_{y'} \end{aligned}$$

Now we substitute these expressions into the PDE. The transformed equation is

$$\begin{aligned} (u_{x'} + 2u_{y'}) + 2(2u_{x'} - u_{y'}) + \left[2\frac{1}{5}(x' + 2y') - \frac{1}{5}(2x' - y')\right]u \\ = 2\left[\frac{1}{5}(x' + 2y')\right]^2 + 3\left[\frac{1}{5}(x' + 2y')\right]\left[\frac{1}{5}(2x' - y')\right] - 2\left[\frac{1}{5}(2x' - y')\right]^2. \end{aligned}$$

Surprisingly, simplifying this gives

$$5u_{x'} + y'u = x'y'.$$

Divide both sides by 5.

$$u_{x'} + \frac{y'}{5}u = \frac{1}{5}x'y'$$

This is a first-order linear inhomogeneous equation. To solve this equation for u, multiply both sides by the integrating factor,

$$I(x') = e^{\int^{x'} \frac{y'}{5}\,ds} = e^{\frac{x'y'}{5}}.$$

$$e^{\frac{x'y'}{5}}u_{x'} + \frac{y'}{5}e^{\frac{x'y'}{5}}u = \frac{1}{5}x'y'e^{\frac{x'y'}{5}}$$

Now write the left side as the x'-derivative of Iu.

$$\frac{\partial}{\partial x'}\left(e^{\frac{x'y'}{5}}u\right) = \frac{1}{5}x'y'e^{\frac{x'y'}{5}}$$

Integrate both sides partially with respect to x'.

$$e^{\frac{x'y'}{5}}u = \int^{x'} \frac{1}{5}sy'e^{\frac{sy'}{5}}\,ds + f(y'),$$

where f is an arbitrary function of y'.

$$e^{\frac{x'y'}{5}}u = e^{\frac{x'y'}{5}}\left(x' - \frac{5}{y'}\right) + f(y')$$

$$u(x', y') = x' - \frac{5}{y'} + e^{-\frac{x'y'}{5}}f(y')$$

All that's left to do now is to plug back in the original variables.

$$u(x, y) = (x + 2y) - \frac{5}{(2x - y)} + e^{-\frac{(x+2y)(2x-y)}{5}}f(2x - y)$$

Therefore,

$$u(x, y) = x + 2y + \frac{5}{y - 2x} + e^{\frac{-2x^2-3xy+2y^2}{5}}f(2x - y).$$

We can check that this is the solution to the PDE.

$$u_x = 1 + \frac{10}{(y - 2x)^2} + \frac{1}{5}e^{\frac{-2x^2-3xy+2y^2}{5}}\left[-(4x + 3y)f + 10f'\right]$$

$$u_y = 2 - \frac{5}{(y - 2x)^2} + \frac{1}{5}e^{\frac{-2x^2-3xy+2y^2}{5}}\left[(-3x + 4y)f - 5f'\right]$$

Hence,

$$\begin{aligned}
u_x + 2u_y + (2x - y)u &= 1 + \cancel{\frac{10}{(y - 2x)^2}} + \frac{1}{5}e^{\frac{-2x^2-3xy+2y^2}{5}}\left[-(4x + 3y)f + \cancel{10f'}\right] \\
&\quad + 4 - \cancel{\frac{10}{(y - 2x)^2}} + \frac{2}{5}e^{\frac{-2x^2-3xy+2y^2}{5}}\left[(-3x + 4y)f - \cancel{5f'}\right] \\
&\quad + (2x - y)\left(x + 2y + \frac{5}{y - 2x} + e^{\frac{-2x^2-3xy+2y^2}{5}}f\right) \\
u_x + 2u_y + (2x - y)u &= \cancel{5} + \frac{1}{5}e^{\frac{-2x^2-3xy+2y^2}{5}}(-10x + 5y)f + (2x - y)(x + 2y) - \cancel{5} \\
&\qquad + (2x - y)e^{\frac{-2x^2-3xy+2y^2}{5}}f \\
u_x + 2u_y + (2x - y)u &= -\cancel{(2x - y)e^{\frac{-2x^2-3xy+2y^2}{5}}}f + (2x - y)(x + 2y) + \cancel{(2x - y)e^{\frac{-2x^2-3xy+2y^2}{5}}}f \\
u_x + 2u_y + (2x - y)u &= 2x^2 + 3xy - 2y^2,
\end{aligned}$$

which means that this is the solution to the PDE.

The Geometric Method: Characteristic Curves

Suppose that we wish to solve the PDE using the method of characteristics. We know that it collapses to an ODE

$$\frac{du}{dx} + (2x - y)u = 2x^2 + 3xy - 2y^2 \tag{1}$$

on the curves in the xy-plane defined by

$$\frac{dy}{dx} = 2, \quad y(\xi, 0) = \xi, . \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (2). (2) implies that y is a function of x, so before we can solve (1), we must solve (2) first. Integrating (2) yields

$$y = 2x + \xi. \tag{3}$$

Plugging this form for y into (1), we get

$$\frac{du}{dx} + [2x - (2x + \xi)]u = 2x^2 + 3x(2x + \xi) - 2(2x + \xi)^2,$$

which simplifies to

$$\frac{du}{dx} - \xi u = -\xi(5x + 2\xi).$$

This is a linear inhomogeneous differential equation for u. That means we have to multiply both sides by an integrating factor in order to solve for u.

$$I(x) = e^{\int^x -\xi\, ds} = e^{-\xi x}$$

$$e^{-\xi x}\frac{du}{dx} - \xi e^{-\xi x} u = -\xi(5x + 2\xi)e^{-\xi x}$$

The left side is just the x-derivative of Iu.

$$\frac{d}{dx}\left(e^{-\xi x} u\right) = -\xi(5x + 2\xi)e^{-\xi x}$$

Now partially integrate both sides with respect to x.

$$e^{-\xi x} u = \int^x -\xi(5s + 2\xi)e^{-\xi s}\, ds + g(\xi),$$

where g is an arbitrary function of ξ.

$$e^{-\xi x} u = \frac{1}{\xi} e^{-\xi x}(5 + 5\xi x + 2\xi^2) + g(\xi)$$

$$u(x, \xi) = \frac{1}{\xi}(5 + 5\xi x + 2\xi^2) + e^{\xi x} g(\xi)$$

Solving (3) for ξ gives

$$\xi = y - 2x.$$

Plugging this in to $u(x, \xi)$ yields the solution, $u(x, y)$.

$$u(x, y) = \frac{1}{y - 2x}[5 + 5(y - 2x)x + 2(y - 2x)^2] + e^{(y-2x)x} g(y - 2x)$$

Simplifying this gives

$$u(x,y) = x + 2y + \frac{5}{y-2x} + e^{x(y-2x)} g(y-2x).$$

We can check to see whether this is the solution to the PDE.

$$\begin{aligned} u_x &= 1 + \frac{10}{(y-2x)^2} - e^{x(y-2x)} \left[(4x-y)g + 2g'\right] \\ u_y &= 2 - \frac{5}{(y-2x)^2} + e^{x(y-2x)} \left(xg + g'\right) \end{aligned}$$

Hence,

$$\begin{aligned} u_x + 2u_y + (2x-y)u &= 1 + \cancel{\frac{10}{(y-2x)^2}} - e^{x(y-2x)} \left[(4x-y)g + \cancel{2g'}\right] \\ &\quad + 4 - \cancel{\frac{10}{(y-2x)^2}} + 2e^{x(y-2x)} \left[xg + \cancel{g'}\right] \\ &\quad + (2x-y)\left(x + 2y + \frac{5}{y-2x} + e^{x(y-2x)} g\right) \\ u_x + 2u_y + (2x-y)u &= \cancel{5} + e^{x(y-2x)}(-2x+y)g + (2x-y)(x+2y) - \cancel{5} + (2x-y)e^{x(y-2x)} g \\ u_x + 2u_y + (2x-y)u &= -\cancel{(2x-y)e^{x(y-2x)}} g + (2x-y)(x+2y) + \cancel{(2x-y)e^{x(y-2x)}} g \\ u_x + 2u_y + (2x-y)u &= 2x^2 + 3xy - 2y^2, \end{aligned}$$

which means that this is the solution to the PDE.

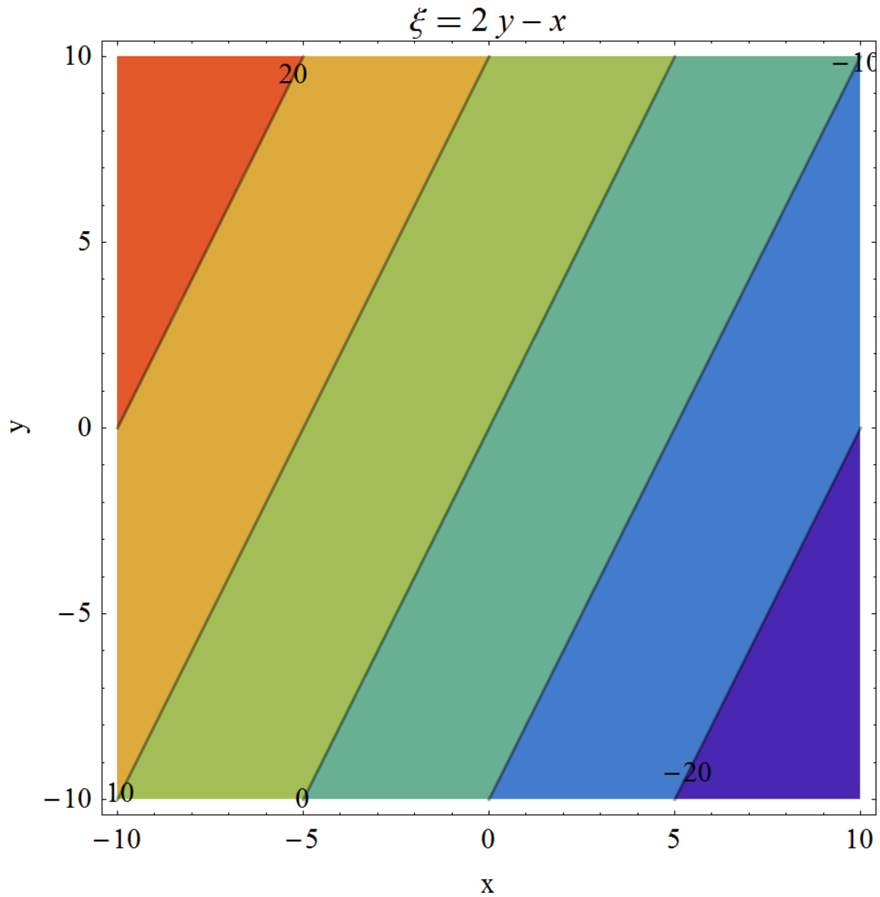

Figure 1: Plot of the characteristic curves in the xy-plane for $-10 < x < 10$ and $-10 < y < 10$.