Exercise 6

Consider heat flow in a long circular cylinder where the temperature depends only on \( t \) and on the distance \( r \) to the axis of the cylinder. Here \( r = \sqrt{x^2 + y^2} \) is the cylindrical coordinate. From the three-dimensional heat equation derive the equation \( u_t = k (u_{rr} + u_r/r) \).

Solution

The three-dimensional heat equation is given as

\[
\rho c u_t = \nabla \cdot (\kappa \nabla u).
\]

If we assume the circular cylinder is homogeneous, then \( \kappa \) does not depend on the spatial coordinates and can be pulled out in front.

\[
c \rho u_t = \kappa \nabla \cdot \nabla u = \kappa \nabla^2 u
\]

Thus,

\[
u_t = k \nabla^2 u, \tag{1}\]

where \( k = \kappa/c \rho \). In cylindrical coordinates, the laplacian operator is defined as follows.

\[
\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}
\]

But since the only spatial coordinate \( u \) depends on is \( r \), \( \partial^2 u/\partial \theta^2 = 0 \) and \( \partial^2 u/\partial z^2 = 0 \).

\[
\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2}
\]

The laplacian simplifies to

\[
\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right).
\]

Substituting this expression for \( \nabla^2 u \) into (1) gives us

\[
u_t = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \right]
\]

\[
u_t = \frac{k}{r} \left( \frac{\partial u}{\partial r} + r \frac{\partial^2 u}{\partial r^2} \right)
\]

\[
u_t = k \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right).
\]

Therefore,

\[
u_t = k \left( u_{rr} + \frac{1}{r} u_r \right).
\]