Exercise 9

This is an exercise on the divergence theorem

$$\iiint_D \nabla \cdot \mathbf{F}\, d\mathbf{x} = \iint_{\text{bdy D}} \mathbf{F} \cdot \mathbf{n}\, dS,$$

valid for any bounded domain D in space with boundary surface bdy D and unit outward normal vector $\mathbf{n}$. If you never learned it, see Section A.3. It is crucial that D be bounded. As an exercise, verify it in the following case by calculating both sides separately: $\mathbf{F} = r^2\mathbf{x}$, $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r^2 = x^2 + y^2 + z^2$, and $D =$ the ball of radius a and center at the origin.

Solution

Calculation of the Left-Hand Side

The aim is to calculate

$$\iiint_D \nabla \cdot \mathbf{F}\, d\mathbf{x}.$$

Start by writing $\mathbf{F}$ in a form we can work with.

$$\begin{aligned}
\mathbf{F} &= r^2\mathbf{x} \\
\mathbf{F} &= r^2x\hat{\mathbf{x}} + r^2y\hat{\mathbf{y}} + r^2z\hat{\mathbf{z}} \\
\mathbf{F} &= \left(x^2+y^2+z^2\right)x\hat{\mathbf{x}} + \left(x^2+y^2+z^2\right)y\hat{\mathbf{y}} + \left(x^2+y^2+z^2\right)z\hat{\mathbf{z}} \\
\mathbf{F} &= \left(x^3+xy^2+xz^2\right)\hat{\mathbf{x}} + \left(x^2y+y^3+yz^2\right)\hat{\mathbf{y}} + \left(x^2z+y^2z+z^3\right)\hat{\mathbf{z}}
\end{aligned}$$

In cartesian coordinates, the divergence of $\mathbf{F}$ is defined as follows.

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\
\nabla \cdot \mathbf{F} &= \left(3x^2+y^2+z^2\right) + \left(x^2+3y^2+z^2\right) + \left(x^2+y^2+3z^2\right) \\
\nabla \cdot \mathbf{F} &= 5x^2+5y^2+5z^2 \\
\nabla \cdot \mathbf{F} &= 5r^2
\end{aligned}$$

So we have to integrate $5r^2$ over the volume of the ball. Because this is just a function of r, the triple integral can be reduced to a single integral.

$$\iiint_D \nabla \cdot \mathbf{F}\, d\mathbf{x} = \int_V 5r^2\, dV = \int_0^a 5r^2 \cdot 4\pi r^2\, dr = 20\pi \int_0^a r^4\, dr = 4\pi a^5$$

Therefore, the left-hand side is

$$\iiint_D \nabla \cdot \mathbf{F}\, d\mathbf{x} = 4\pi a^5.$$

Calculation of the Right-Hand Side

The aim is to calculate

$$\iint_{\text{bdy D}} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS. \tag{1}$$

Because the region in question is a sphere, the outward unit vector perpendicular to the surface is in the radial direction. If a point is located at (x, y, z), the unit vector in this direction is

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle = \frac{1}{r} \langle x, y, z \rangle.$$

And just to repeat,

$$\mathbf{F} = r^2 \mathbf{x} = r^2 \langle x, y, z \rangle.$$

Because we are evaluating this integral over the surface of the ball, the radius r is equal to a. If we plug these expressions into (1), we get

$$\iint_{\text{bdy D}} a^2 \langle x, y, z \rangle \cdot \frac{1}{a} \langle x, y, z \rangle \, dS = \iint_{\text{bdy D}} a \left(x^2 + y^2 + z^2 \right) \, dS.$$

At any point on the sphere, $x^2 + y^2 + z^2 = a^2$. Hence,

$$\iint_{\text{bdy D}} a^3 \, dS = a^3 \iint_{\text{bdy D}} dS = a^3 \cdot 4\pi a^2 = 4\pi a^5.$$

Note that $\iint dS$ is just the surface area of the ball. Therefore, the right-hand side is

$$\iint_{\text{bdy D}} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 4\pi a^5.$$

Because both sides are equal, the divergence theorem is verified.