Exercise 10

If $\mathbf{f}(\mathbf{x})$ is continuous and $|\mathbf{f}(\mathbf{x})| \leq 1/(|\mathbf{x}|^3 + 1)$ for all $\mathbf{x}$, show that

$$\iiint_{\text{all space}} \nabla \cdot \mathbf{f}\, d\mathbf{x} = 0.$$

(*Hint:* Take D to be a large ball, apply the divergence theorem, and let its radius tend to infinity.)

Solution[1]

Consider the volume integral,

$$\iiint_D \nabla \cdot \mathbf{f}\, dV,$$

where D is a ball in xyz-space centered at the origin with radius R. The aim in this exercise is to show that

$$\lim_{R\to\infty} \iiint_D \nabla \cdot \mathbf{f}\, dV = 0.$$

Apply the divergence theorem to turn the volume integral into a surface integral over the ball's boundary bdy D.

$$\iiint_D \nabla \cdot \mathbf{f}\, dV = \oiint_{\text{bdy } D} \mathbf{f} \cdot \hat{\mathbf{n}}\, dS$$

Here $\hat{\mathbf{n}}$ is a (radial) unit vector normal to the ball's surface. Take the absolute value of both sides of the equation and simplify the right side.

$$\begin{aligned}
\left| \iiint_D \nabla \cdot \mathbf{f}\, dV \right| &= \left| \oiint_{\text{bdy } D} \mathbf{f} \cdot \hat{\mathbf{n}}\, dS \right| \\
&\leq \oiint_{\text{bdy } D} |\mathbf{f} \cdot \hat{\mathbf{n}}|\, dS \\
&\leq \oiint_{\text{bdy } D} |\mathbf{f}||\hat{\mathbf{n}}|\, dS \\
&= \oiint_{\text{bdy } D} |\mathbf{f}|\, dS \\
&\leq \oiint_{\text{bdy } D} \frac{1}{R^3 + 1}\, dS \\
&= \int_0^{\pi} \int_0^{2\pi} \frac{1}{R^3 + 1} (R^2 \sin\theta\, d\phi\, d\theta) \\
&= \frac{R^2}{R^3 + 1} \left(\int_0^{\pi} \sin\theta\, d\theta \right) \left(\int_0^{2\pi} d\phi \right) \\
&= \frac{R^2}{R^3 + 1} (2)(2\pi)
\end{aligned}$$

[1] Thank you, J. V. Winkle, for the correction.

In evaluating the surface integral, spherical coordinates were used with θ representing the angle from the polar axis. Take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty} \left| \iiint_D \nabla \cdot \mathbf{f} \, dV \right| \le \lim_{R\to\infty} \frac{R^2}{R^3+1}(4\pi) = \lim_{R\to\infty} \frac{4\pi}{R + \frac{1}{R^2}}$$

The denominator blows up as $R \to \infty$, so the limit on the right side is zero.

$$\lim_{R\to\infty} \left| \iiint_D \nabla \cdot \mathbf{f} \, dV \right| \le 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty} \left| \iiint_D \nabla \cdot \mathbf{f} \, dV \right| = 0$$

The only number with a magnitude of zero is zero.

$$\lim_{R\to\infty} \iiint_D \nabla \cdot \mathbf{f} \, dV = 0$$

Therefore,

$$\iiint_{\text{all space}} \nabla \cdot \mathbf{f} \, d\mathbf{x} = 0.$$