Exercise 11

If $\operatorname{curl}\mathbf{v} = \mathbf{0}$ in all of three-dimensional space, show that there exists a scalar function $\phi(x, y, z)$ such that $\mathbf{v} = \operatorname{grad}\phi$.

Solution

The proof can be broken down into three steps.

1. Show that for any oriented simple closed curve C,
$$\oint_C \mathbf{v} \cdot d\mathbf{r} = 0.$$
2. Show that for any two oriented simple closed curves C_1 and C_2 that have the same endpoints,
$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r} = \int_{C_2} \mathbf{v} \cdot d\mathbf{r}.$$
3. Show that there exists a scalar function $\phi(x, y, z)$ such that
$$\mathbf{v} = \operatorname{grad}\phi.$$

Step 1

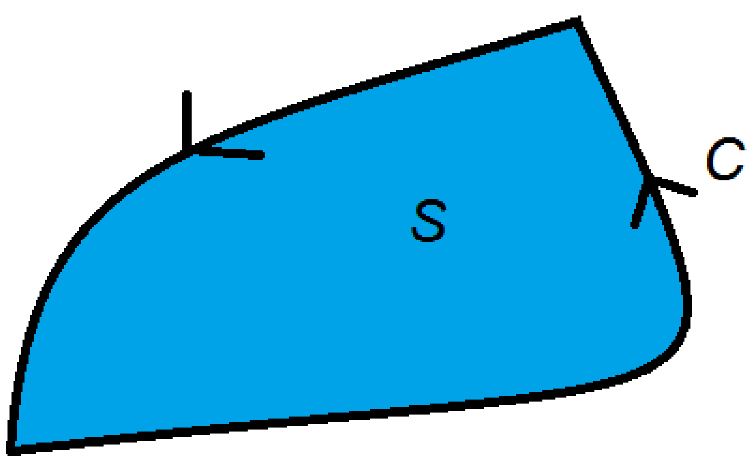

Figure 1: A surface S bounded by an oriented simple closed curve C in $\mathbb{R}^3$.

A simple curve is one that does not cross itself at any point, a closed curve is one where the initial point coincides with the final point, and an oriented curve is one that is traversed in the counterclockwise sense. Stokes's theorem tells us that if we have a surface S that is bounded by an oriented simple closed closed curve C in $\mathbb{R}^3$, then

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{v} \cdot d\mathbf{S}.$$

But since $\operatorname{curl}\mathbf{v} = \mathbf{0}$,

$$\iint_S \operatorname{curl}\mathbf{v}\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0.$$

And Stokes's theorem implies that

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = 0.$$

Step 1 is proven.

Step 2

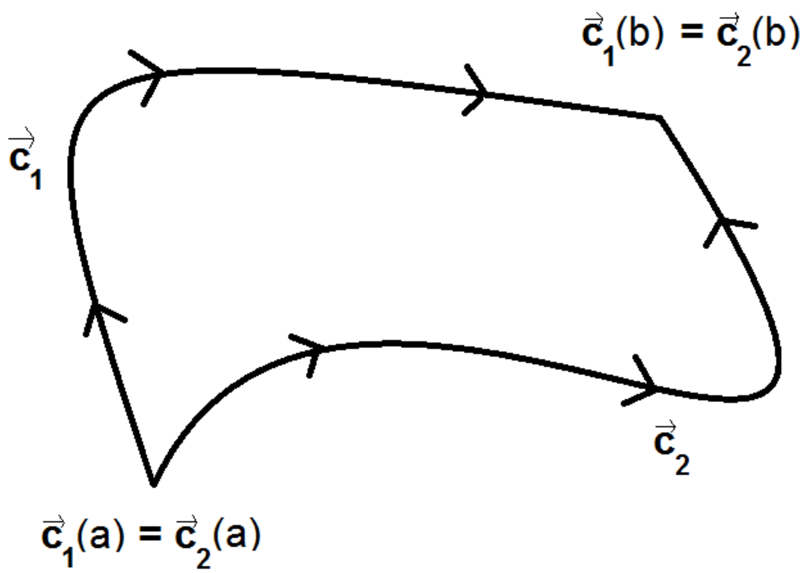

Figure 2: Two paths to get from a to b in space and their parametric representations.

Suppose that C_1 and C_2 are the paths taken to get from an initial point a to a final point b in space and that $\mathbf{c}_1$ and $\mathbf{c}_2$ are the respective parameterizations of these curves. In order to construct an oriented simple closed curve from these two parameterizations, take the negative of $\mathbf{c}_1$ and add it to $\mathbf{c}_2$.

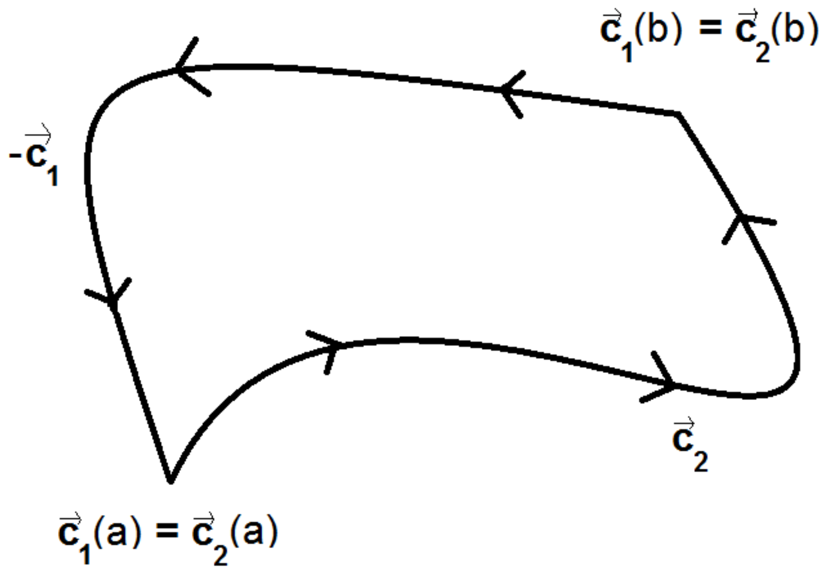

Figure 3: Switching to $-\mathbf{c}_1$ gives an oriented simple closed curve.

If we start at a and integrate $\mathbf{v}$ counterclockwise around this curve coming back to a, we get

$$\begin{aligned}\oint_C \mathbf{v}\cdot d\mathbf{r} &= \int_{C_2} \mathbf{v}\cdot d\mathbf{r} + \int_{-C_1} \mathbf{v}\cdot d\mathbf{r} \\ &= \int_{C_2} \mathbf{v}\cdot d\mathbf{r} - \int_{C_1} \mathbf{v}\cdot d\mathbf{r}.\end{aligned}$$

But we know from step 1 that $\oint_C \mathbf{v}\cdot d\mathbf{r}$ has to equal 0 for an oriented simple closed curve. Therefore,

$$\int_{C_2} \mathbf{v}\cdot d\mathbf{r} - \int_{C_1} \mathbf{v}\cdot d\mathbf{r} = 0$$

$$\int_{C_1} \mathbf{v}\cdot d\mathbf{r} = \int_{C_2} \mathbf{v}\cdot d\mathbf{r}.$$

Step 2 is proven.

Step 3

Let C be the path from the origin to (x, y, z) and let $\mathbf{c}$ be the parameterization of this path. We define the scalar (potential) function $\phi(x, y, z)$ as follows.

$$\phi(x, y, z) = \int_C \mathbf{v}\cdot d\mathbf{r}$$

Based on step 2, it should not matter how we get to (x, y, z) from the origin, so we have the freedom to choose whatever parameterization we want. Assume that $\mathbf{v} = \langle v_x, v_y, v_z\rangle$.

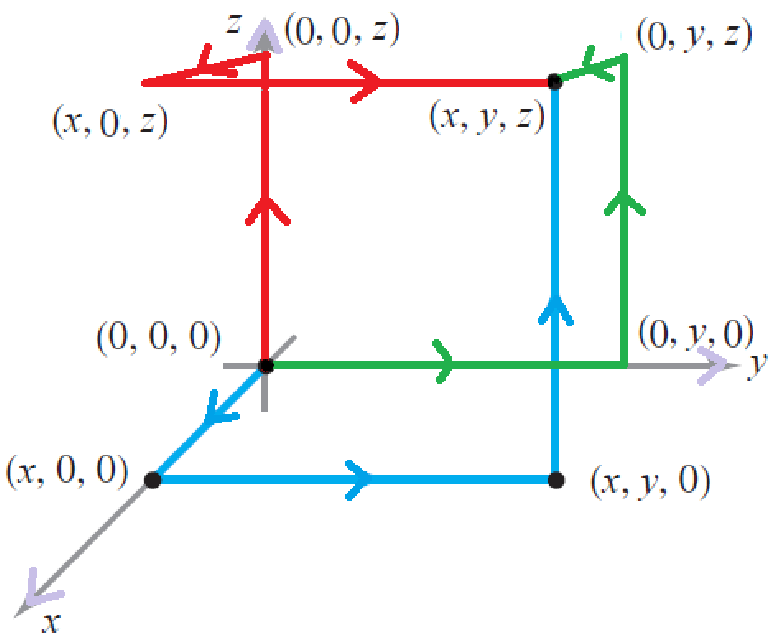

Figure 4: Three paths from the origin to (x, y, z).

Along the green path,

$$\phi(x,y,z) = \int_0^y v_y(0,t,0)\,dt + \int_0^z v_z(0,y,t)\,dt + \int_0^x v_x(t,y,z)\,dt. \tag{1}$$

Along the red path,

$$\phi(x,y,z) = \int_0^z v_z(0,0,t)\,dt + \int_0^x v_x(t,0,z)\,dt + \int_0^y v_y(x,t,z)\,dt. \tag{2}$$

Along the blue path,

$$\phi(x,y,z) = \int_0^x v_x(t,0,0)\,dt + \int_0^y v_y(x,t,0)\,dt + \int_0^z v_z(x,y,t)\,dt. \tag{3}$$

If we differentiate both sides of (1) with respect to x, both sides of (2) with respect to y, and both sides of (3) with respect to z, we get

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \underbrace{\frac{\partial}{\partial x}\int_0^y v_y(0,t,0)\,dt}_{=0} + \underbrace{\frac{\partial}{\partial x}\int_0^z v_z(0,y,t)\,dt}_{=0} + \frac{\partial}{\partial x}\int_0^x v_x(t,y,z)\,dt \\
\frac{\partial\phi}{\partial y} &= \underbrace{\frac{\partial}{\partial y}\int_0^z v_z(0,0,t)\,dt}_{=0} + \underbrace{\frac{\partial}{\partial y}\int_0^x v_x(t,0,z)\,dt}_{=0} + \frac{\partial}{\partial y}\int_0^y v_y(x,t,z)\,dt \\
\frac{\partial\phi}{\partial z} &= \underbrace{\frac{\partial}{\partial z}\int_0^x v_x(t,0,0)\,dt}_{=0} + \underbrace{\frac{\partial}{\partial z}\int_0^y v_y(x,t,0)\,dt}_{=0} + \frac{\partial}{\partial z}\int_0^z v_z(x,y,t)\,dt.
\end{aligned}$$

According to the fundamental theorem of calculus,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Thus,

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= v_x \\ \frac{\partial \phi}{\partial y} &= v_y \\ \frac{\partial \phi}{\partial z} &= v_z.\end{aligned}$$

This means that

$$\mathbf{v} = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle,$$

that is,

$$\mathbf{v} = \operatorname{grad}\phi.$$

Note that our initial point didn't have to be the origin. If we started at some arbitrary point, the same argument would hold. Also note that, in constructing the three colored paths, it's important that the final segment to (x, y, z) be in the positive direction of the variable we wish to differentiate with respect to. Step 3 is proven and this concludes the proof.