Exercise 4

A rod occupying the interval $0 \leq x \leq l$ is subject to the heat source $f(x) = 0$ for $0 < x < \frac{l}{2}$, and $f(x) = H$ for $\frac{l}{2} < x < l$ where $H > 0$. The rod has physical constants $c = \rho = \kappa = 1$, and its ends are kept at zero temperature.

(a) Find the steady-state temperature of the rod.

(b) Which point is the hottest, and what is the temperature there?

Solution

Part (a)

The governing PDE for the temperature in a bar with a heat source is given by

$$T_t = \nabla \cdot (k \nabla T) + f(x,t).$$

The steady state of the temperature is reached as $t \to \infty$. In other words, after a long time has passed, the temperature will no longer vary as a function of time. Hence, $T_t = 0$. Because all constants are unity, the heat source varies only with position, and the rod is one-dimensional, the heat equation reduces to

$$0 = \frac{d^2T}{dx^2} + f(x)$$

in the steady state. That is,

$$\begin{cases}
0 = \frac{d^2T}{dx^2} & 0 < x < \frac{l}{2} \\
0 = \frac{d^2T}{dx^2} + H & \frac{l}{2} < x < l.
\end{cases}$$

These two ODEs can be solved by straightforward integration.

$$T(x) = \begin{cases}
Ax + B & 0 < x < \frac{l}{2} \\
-\frac{1}{2}Hx^2 + Cx + D & \frac{l}{2} < x < l
\end{cases}$$

We're told that the ends of the rod are kept at zero temperature, so the boundary conditions are $T(0, t) = 0$ and $T(l, t) = 0$ for all time. In the steady state, though, we use $T(0) = 0$ and $T(l) = 0$. 

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We have four constants of integration; therefore, we need two more conditions to determine the solution. These two additional conditions come from physical grounds: The temperature and the heat flux must be continuous at the point, \( x = l/2 \), where the heat source changes. Thus,

\[
\lim_{x \to l/2^-} T(x) = \lim_{x \to l/2^+} T(x) \\
\lim_{x \to l/2^-} -\kappa \frac{dT}{dx} = \lim_{x \to l/2^+} -\kappa \frac{dT}{dx}.
\]

These four conditions mean that

\[
T(0) = 0 \quad \rightarrow \quad B = 0 \\
T(l) = 0 \quad \rightarrow \quad D = \frac{1}{2} H l^2 - Cl \\
T \left( \frac{l}{2} \right) = T \left( \frac{l}{2}^+ \right) \quad \rightarrow \quad A l = \frac{l}{8} \left( 3Hl - 4C \right) \\
\frac{dT}{dx} \left( \frac{l}{2}^- \right) = \frac{dT}{dx} \left( \frac{l}{2}^+ \right) \quad \rightarrow \quad A = C - \frac{Hl}{2}.
\]

Solving this system of equations for the constants, plugging in the values, and simplifying, we arrive at the steady state temperature.

\[
T(x) = \begin{cases} 
\frac{Hl}{8} x & 0 < x < \frac{l}{2} \\
-\frac{H}{8} (l - 4x)(l - x) & \frac{l}{2} < x < l 
\end{cases}
\]

Setting \( l = H = 1 \), we can graph this to get an idea of what the temperature looks like in the steady state.

![T](x) vs. \( x \).

Figure 2: Plot of \( T(x) \) vs. \( x \).
Part (b)

The maximum temperature occurs where the first derivative of the steady-state temperature equals zero. Taking the derivative with respect to $x$ of the function defined for $\frac{l}{2} < x < l$ gives

$$H\left(\frac{5l}{8} - x\right).$$

By inspection we can see that if

$$x = \frac{5l}{8},$$

the first derivative vanishes, and that means this is where the hottest temperature occurs in the bar. Evaluating $T(x)$ at this point tells us what the hottest temperature is.

$$T\left(x = \frac{5l}{8}\right) = \frac{9Hl^2}{128}$$

$9/128 \approx 0.0703$ and $5/8 = 0.625$, so the graph is consistent with our conclusions.