Exercise 6

Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/c\rho$ be their diffusion constants. They are welded together so that the temperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature T degrees.

(a) Find the *equilibrium* temperature distribution in the composite rod.

(b) Sketch it as a function of x in case $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$, and $T = 10$. (This exercise requires a lot of elementary algebra, but it's worth it.)

Solution

Part (a)

The governing PDE for the temperature in a bar is given by the heat equation.

$$c\rho u_t = \nabla \cdot (\kappa \nabla u)$$

However, since the bar has different physical properties along different portions, the equation applies to each portion separately.

$$\begin{cases} c\rho u_t = \nabla \cdot (\kappa_1 \nabla u) & 0 < x < L_1 \\ c\rho u_t = \nabla \cdot (\kappa_2 \nabla u) & L_1 < x < L_1 + L_2 \end{cases}$$

Equilibrium (the steady state) is reached after a long time has passed ($t \to \infty$) since the temperature will no longer vary as a function of time. Hence, $u_t = 0$. Because the rod is one-dimensional and κ_1 and κ_2 are constants, the heat equation reduces to

$$\begin{cases} 0 = \dfrac{d^2u}{dx^2} & 0 < x < L_1 \\ 0 = \dfrac{d^2u}{dx^2} & L_1 < x < L_1 + L_2 \end{cases}$$

at equilibrium. These two ODEs can be solved by straightforward integration.

$$u(x) = \begin{cases} Ax + B & 0 < x < L_1 \\ Cx + D & L_1 < x < L_1 + L_2 \end{cases}$$

We're told that the left end of the rod is kept at zero degrees and that the right end of the rod is kept at T degrees, so the boundary conditions are $u(0,t) = 0$ and $u(L_1 + L_2, t) = T$ for all time. At equilibrium, though, we use $u(0) = 0$ and $u(L_1 + L_2) = 0$. We have four constants of integration; therefore, we need two more conditions to determine the solution. These two additional conditions come from physical grounds: The temperature and the heat flux must be continuous at the welding point, $x = L_1$. Thus,

$$\lim_{x \to L_1-} u(x) = \lim_{x \to L_1+} u(x)$$
$$\lim_{x \to L_1-} -\kappa_1 \frac{du}{dx} = \lim_{x \to L_1+} -\kappa_2 \frac{du}{dx}.$$

These four conditions mean that

$$\begin{aligned} u(0) = 0 \quad &\rightarrow \quad B = 0 \\ u(L_1 + L_2) = T \quad &\rightarrow \quad C(L_1 + L_2) + D = T \\ u(L_1-) = u(L_1+) \quad &\rightarrow \quad AL_1 + B = CL_1 + D \\ \kappa_1 \frac{du}{dx}(L_1-) = \kappa_2 \frac{du}{dx}(L_1+) \quad &\rightarrow \quad \kappa_1 A = \kappa_2 C. \end{aligned}$$

Solving this system of equations for the constants, plugging in the values, and simplifying, we arrive at the equilibrium temperature.

$$u(x) = \begin{cases} \dfrac{\kappa_2 T}{\kappa_2 L_1 + \kappa_1 L_2} x & 0 < x < L_1 \\ T - \dfrac{\kappa_1 T}{\kappa_2 L_1 + \kappa_1 L_2}(L_1 + L_2 - x) & L_1 < x < L_1 + L_2 \end{cases}$$

Part (b)

Setting $\kappa_1 = 2$, $\kappa_2 = 1$, $L_1 = 3$, $L_2 = 2$, and $T = 10$, this equation becomes

$$u(x) = \begin{cases} \dfrac{10}{7} x & 0 < x < 3 \\ \dfrac{10}{7}(2x - 3) & 3 < x < 5 \end{cases}.$$

We can graph this to get an idea of what the temperature looks like at equilibrium.

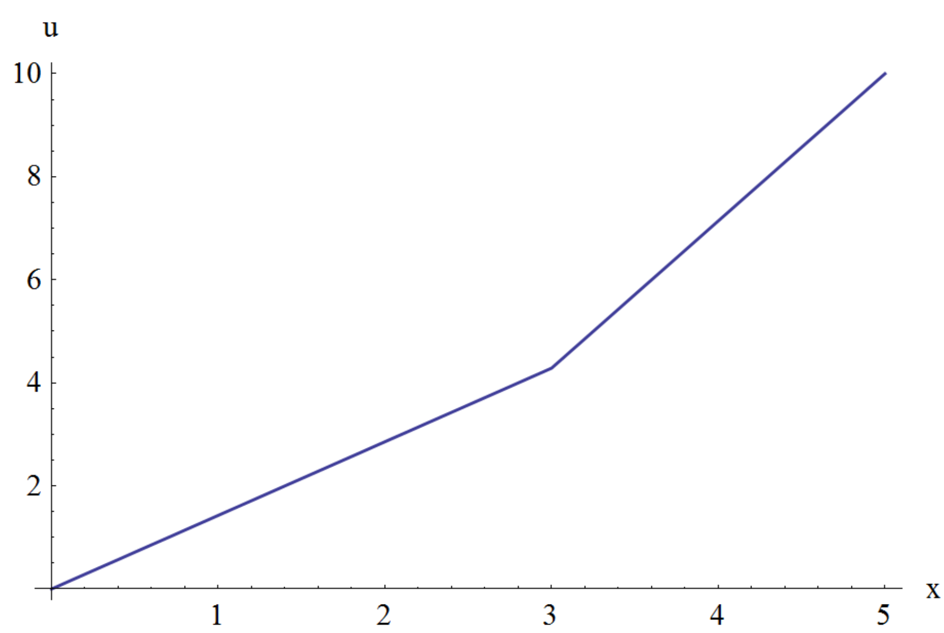

Figure 1: Plot of $u(x)$ vs. x.