Exercise 6

Two homogeneous rods have the same cross section, specific heat $c$, and density $\rho$ but different heat conductivities $\kappa_1$ and $\kappa_2$ and lengths $L_1$ and $L_2$. Let $k_j = \kappa_j/c\rho$ be their diffusion constants. They are welded together so that the temperature $u$ and the heat flux $\kappa u_x$ at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature $T$ degrees.

(a) Find the *equilibrium* temperature distribution in the composite rod.

(b) Sketch it as a function of $x$ in case $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$, and $T = 10$. (This exercise requires a lot of elementary algebra, but it’s worth it.)

**Solution**

**Part (a)**

The governing PDE for the temperature in a bar is given by the heat equation.

$$c\rho u_t = \nabla \cdot (\kappa \nabla u)$$

However, since the bar has different physical properties along different portions, the equation applies to each portion separately.

$$\begin{cases} c\rho u_t = \nabla \cdot (\kappa_1 \nabla u) & 0 < x < L_1 \\ c\rho u_t = \nabla \cdot (\kappa_2 \nabla u) & L_1 < x < L_1 + L_2 \end{cases}$$

Equilibrium (the steady state) is reached after a long time has passed ($t \to \infty$) since the temperature will no longer vary as a function of time. Hence, $u_t = 0$. Because the rod is one-dimensional and $\kappa_1$ and $\kappa_2$ are constants, the heat equation reduces to

$$\begin{cases} 0 = \frac{d^2 u}{dx^2} & 0 < x < L_1 \\ 0 = \frac{d^2 u}{dx^2} & L_1 < x < L_1 + L_2 \end{cases}$$

at equilibrium. These two ODEs can be solved by straightforward integration.

$$u(x) = \begin{cases} Ax + B & 0 < x < L_1 \\ Cx + D & L_1 < x < L_1 + L_2 \end{cases}$$

We’re told that the left end of the rod is kept at zero degrees and that the right end of the rod is kept at $T$ degrees, so the boundary conditions are $u(0, t) = 0$ and $u(L_1 + L_2, t) = T$ for all time. At equilibrium, though, we use $u(0) = 0$ and $u(L_1 + L_2) = 0$. We have four constants of integration; therefore, we need two more conditions to determine the solution. These two additional conditions come from physical grounds: The temperature and the heat flux must be continuous at the welding point, $x = L_1$. Thus,

$$\lim_{x \to L_1^-} u(x) = \lim_{x \to L_1^+} u(x)$$

$$\lim_{x \to L_1^-} -\kappa_1 \frac{du}{dx} = \lim_{x \to L_1^+} -\kappa_2 \frac{du}{dx}.$$
These four conditions mean that

\[
\begin{align*}
  u(0) &= 0 \quad \rightarrow \quad B = 0 \\
  u(L_1 + L_2) &= T \quad \rightarrow \quad C(L_1 + L_2) + D = T \\
  u(L_1 -) &= u(L_1 +) \quad \rightarrow \quad AL_1 + B = CL_1 + D \\
  \kappa_1 \frac{du}{dx}(L_1 -) &= \kappa_2 \frac{du}{dx}(L_1 +) \quad \rightarrow \quad \kappa_1 A = \kappa_2 C.
\end{align*}
\]

Solving this system of equations for the constants, plugging in the values, and simplifying, we arrive at the equilibrium temperature.

\[
 u(x) = \begin{cases}
  \frac{\kappa_2 T}{\kappa_2 L_1 + \kappa_1 L_2} x & 0 < x < L_1 \\
  T - \frac{\kappa_1 T}{\kappa_2 L_1 + \kappa_1 L_2} (L_1 + L_2 - x) & L_1 < x < L_1 + L_2 
\end{cases}
\]

**Part (b)**

Setting \( \kappa_1 = 2, \kappa_2 = 1, L_1 = 3, L_2 = 2, \) and \( T = 10, \) this equation becomes

\[
 u(x) = \begin{cases}
  \frac{10}{7} x & 0 < x < 3 \\
  \frac{10}{7} (2x - 3) & 3 < x < 5
\end{cases}
\]

We can graph this to get an idea of what the temperature looks like at equilibrium.

![Figure 1: Plot of \( u(x) \) vs. \( x \).](www.stemjock.com)