Exercise 7

In linearized gas dynamics (sound), verify the following.

(a) If \( \text{curl } \mathbf{v} = 0 \) at \( t = 0 \), then \( \text{curl } \mathbf{v} = 0 \) at all later times.

(b) Each component of \( \mathbf{v} \) and \( \rho \) satisfies the wave equation.

Solution

The linearized equations for the velocity \( \mathbf{v} \) and density \( \rho \) are given as

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + \frac{c_0^2}{\rho_0} \text{grad } \rho &= 0 \\
\frac{\partial \rho}{\partial t} + \rho_0 \text{div } \mathbf{v} &= 0.
\end{align*}
\]  

(1)  
(2)

Part (a)

We want to show that \( \text{curl } \mathbf{v} \) is constant for all time, that is,

\[
\frac{d}{dt} \text{curl } \mathbf{v} = 0.
\]

Solve (1) for \( \frac{\partial \mathbf{v}}{\partial t} \) and substitute it.

\[
\frac{d}{dt} \text{curl } \mathbf{v} = \text{curl } \frac{\partial \mathbf{v}}{\partial t}
\]

\[
\frac{d}{dt} \text{curl } \mathbf{v} = \text{curl } \left( -\frac{c_0^2}{\rho_0} \text{grad } \rho \right)
\]

But since the curl of the gradient of any vector function is \( 0 \), the right side vanishes.

\[
\frac{d}{dt} \text{curl } \mathbf{v} = 0
\]

Therefore, \( \text{curl } \mathbf{v} \) remains constant in time. If it is equal to \( 0 \) at \( t = 0 \), then it is equal to \( 0 \) at all later times.
Part (b)

We start by differentiating both sides of (1) and (2) with respect to time.

\[
\begin{align*}
\frac{\partial^2 v}{\partial t^2} + \frac{c_0^2}{\rho_0} \frac{d}{dt} \text{grad} \rho &= 0 \\
\frac{\partial^2 \rho}{\partial t^2} + \rho_0 \frac{d}{dt} \text{div} \mathbf{v} &= 0 \\
\frac{\partial^2 v}{\partial t^2} + \frac{c_0^2}{\rho_0} \text{grad} \frac{\partial \rho}{\partial t} &= 0 \\
\frac{\partial^2 \rho}{\partial t^2} + \rho_0 \frac{\partial \mathbf{v}}{\partial t} &= 0
\end{align*}
\]

Now substitute (2) into the first equation and (1) into the second equation.

\[
\begin{align*}
\frac{\partial^2 v}{\partial t^2} + \frac{c_0^2}{\rho_0} \text{grad} (-\rho_0 \text{div} \mathbf{v}) &= 0 \\
\frac{\partial^2 \rho}{\partial t^2} + \rho_0 \text{div} \left(-\frac{c_0^2}{\rho_0} \text{grad} \rho \right) &= 0 \\
\frac{\partial^2 v}{\partial t^2} - c_0^2 \text{grad} (\text{div} \mathbf{v}) &= 0 \\
\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \text{div} (\text{grad} \rho) &= 0 \\
\frac{\partial^2 v}{\partial t^2} - c_0^2 \Delta \mathbf{v} &= 0 \\
\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \Delta \rho &= 0
\end{align*}
\]

The first equation implies that

\[
\begin{align*}
\frac{\partial^2 v_x}{\partial t^2} - c_0^2 \Delta v_x &= 0 \\
\frac{\partial^2 v_y}{\partial t^2} - c_0^2 \Delta v_y &= 0 \\
\frac{\partial^2 v_z}{\partial t^2} - c_0^2 \Delta v_z &= 0
\end{align*}
\]

Therefore, each component of $\mathbf{v}$ and $\rho$ satisfy the wave equation.