Exercise 1

Consider the problem

$$\frac{d^2u}{dx^2} + u = 0$$

$$u(0) = 0 \quad \text{and} \quad u(L) = 0,$$

consisting of an ODE and a pair of boundary conditions. Clearly, the function $u(x) \equiv 0$ is a solution. Is this solution *unique, or not*? Does the answer depend on L?

Solution

This is a linear ODE with constant coefficients, so the solution will be of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \rightarrow \quad \frac{du}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2u}{dx^2} = r^2e^{rx}$$

Substituting these expressions into the equation gives us

$$r^2e^{rx} + e^{rx} = 0.$$

Dividing both sides by e^{rx} yields a polynomial in r that we can solve.

$$r^2 + 1 = 0 \quad \rightarrow \quad r = \{\pm i\}$$

Thus,

$$u(x) = Ae^{ix} + Be^{-ix}.$$

Euler's formula is

$$e^{ix} = \cos x + i \sin x,$$

so an equivalent way to write $u(x)$ is

$$u(x) = C \cos x + D \sin x.$$

This form of the solution is preferable since there are no imaginary numbers. We now determine the constants, C and D, by using the boundary conditions.

$$u(0) = 0 \quad \rightarrow \quad C = 0$$

$$u(L) = 0 \quad \rightarrow \quad D \sin L = 0$$

This last equation tells us that either D or $\sin L$ is zero. If D is zero, then we just get the trivial solution, $u(x) = 0$. The more interesting case is if $\sin L = 0$. This occurs if L is some multiple of π, that is, $L = n\pi$ for some integer n. Therefore, the solution, $u(x) = 0$, is unique, provided that L is not some multiple of π. If L is some multiple of π, then $u(x) = 0$ is not unique because both $u(x) = 0$ and $u(x) = D \sin x$ satisfy the boundary value problem. (D is an arbitrary constant.)