Exercise 4

Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D.$$

(a) What can we surely add to any solution to get another solution? So we don't have uniqueness.

(b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = 0$$

is a necessary condition for the Neumann problem to have a solution.

(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

Solution

Part (a)

We can add a constant to any solution to get another solution. Thus, the solution is not unique.

Part (b)

Integrate both sides of the PDE over the volume, D.

$$\iiint_D \Delta u\, dV = \iiint_D f\, dV$$

The laplacian of u can be written as $\Delta u = \nabla^2 u = \nabla \cdot \nabla u$.

$$\iiint_D \nabla \cdot (\nabla u)\, dV = \iiint_D f\, dV \tag{1}$$

Recall from calculus that the divergence theorem says if we have a vector field $\mathbf{F}$, then

$$\iiint_D \nabla \cdot \mathbf{F}\, d\mathbf{x} = \iint_{\text{bdy } D} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS,$$

where $\hat{\mathbf{n}}$ is an outward unit vector perpendicular to the surface. Applying this to the left side of (1), we get

$$\iint_{\text{bdy } D} \nabla u \cdot \hat{\mathbf{n}}\, dS = \iiint_D f\, dV.$$

The component of the gradient in the normal direction is defined as $\nabla u \cdot \hat{\mathbf{n}} = \partial u/\partial n$.

$$\iint\limits_{\text{bdy } D} \frac{\partial u}{\partial n}\, dS = \iiint\limits_{D} f\, dV$$

Since $\partial u/\partial n = 0$ on the boundary of D, the left side equals zero.

$$\iint\limits_{\text{bdy } D} 0\, dS = \iiint\limits_{D} f\, dV$$

Therefore,

$$\iiint\limits_{D} f\, dV = 0.$$

Part (c)

Yes, physical interpretations can be given for both heat flow and diffusion for parts (a) and (b). The fact that we can add any constant to the solution means that the body, D, can be at any temperature. The caveat, as implied by the boundary condition, is that no heat can enter or leave the boundary of D. Because of this, the heat source, f, must average to 0 over the volume of D in order for the temperature to remain finite. That is, if heat is generated in one part of the body, there must be another part of the body where that heat is lost. If f averages to less than 0, then there is a net heat loss in D, and the temperature will fall indefinitely. Conversely, if f averages to more than 0, then there is a net heat gain, and the temperature will rise indefinitely.

There is an analogous interpretation for diffusion. The fact that we can add any constant to the solution means that the body, D, can contain any amount of mass. The caveat, as implied by the boundary condition, is that no flux of mass occurs across the boundary of D. Because of this, the mass source, f, must average to 0 over the volume of D in order for the mass to remain finite. That is, if mass is generated in one part of the body, there must be another part of the body where that mass is lost. If f averages to less than 0, then there is a net mass loss in D, and the mass will fall indefinitely. Conversely, if f averages to more than 0, then there is a net mass gain, and the mass will rise indefinitely. In other words, physically impossible things happen if f does not average to 0 over D.