Exercise 2

Consider the problem
\[ u''(x) + u'(x) = f(x) \]
\[ u'(0) = u(0) = \frac{1}{2}[u'(l) + u(l)] , \]
with \( f(x) \) a given function.

(a) Is the solution unique? Explain.

(b) Does a solution necessarily exist, or is there a condition that \( f(x) \) must satisfy for existence? Explain.

Solution

Part (a)

Suppose there are two solutions to this boundary value problem, \( u_1 \) and \( u_2 \). Then they both satisfy the ODE.

\[ u''_1(x) + u'_1(x) = f(x) \]
\[ u''_2(x) + u'_2(x) = f(x) \]

If there is a unique solution to the problem, then \( u_1 \) and \( u_2 \) must be equal. Subtract the second equation from the first.

\[ u''_1 - u''_2 + u'_1 - u'_2 = 0 \]
\[ (u_1 - u_2)'' + (u_1 - u_2)' = 0 \]

Make the substitution, \( w = u_1 - u_2 \).

\[ w'' + w' = 0 \]

This is a linear ODE with constant coefficients, so the solution will be of the form, \( w = e^{rx} \).

\[ w = e^{rx} \quad \rightarrow \quad \frac{dw}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2w}{dx^2} = r^2e^{rx} \]

Substituting these expressions into the equation gives us

\[ r^2e^{rx} + re^{rx} = 0. \]

Dividing both sides by \( e^{rx} \) yields a polynomial in \( r \) that we can solve.

\[ r^2 + r = 0 \]
\[ r(r + 1) = 0 \]
\[ r = \{-1, 0\} \]

Thus,

\[ w(x) = Ae^{-x} + B. \]

Because \( w = u_1 - u_2 \neq 0 \), the solution to the ODE is not unique.
Part (b)

Start off by integrating both sides of the ODE from 0 to $l$.

$$u'' + u' = f$$

$$\int_{0}^{l} (u'' + u') \, dx = \int_{0}^{l} f(x) \, dx$$

$$\int_{0}^{l} u'' \, dx + \int_{0}^{l} u' \, dx = \int_{0}^{l} f(x) \, dx$$

$$u'|_{0}^{l} + u|_{0}^{l} = \int_{0}^{l} f(x) \, dx$$

$$u'(l) - u'(0) + u(l) - u(0) = \int_{0}^{l} f(x) \, dx$$

The boundary conditions are

$$u(0) = u'(0) = \frac{1}{2} [u'(l) + u(l)],$$

so if we plug these in to the left side of the equation, we get

$$u'(l) - \frac{1}{2} [u'(l) + u(l)] + u(l) - \frac{1}{2} [u'(l) + u(l)] = \int_{0}^{l} f(x) \, dx$$

$$y'(l) - y'(0) + \hat{u}(l) - \hat{u}(0) = \int_{0}^{l} f(x) \, dx.$$

Therefore, in order for a solution to exist, $f(x)$ must satisfy the following condition.

$$\int_{0}^{l} f(x) \, dx = 0$$

That is, the average of $f$ over the length must be 0 for the solution to exist.