Exercise 3

Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

Solution

The ODE can be solved simply by integrating.

$$u(x) = Ax + B$$

Now we use the first boundary condition.

$$u'(0) + ku(0) = 0 \quad \rightarrow \quad A + kB = 0 \tag{1}$$

Case $+$

Here we use the second ($+$) boundary condition.

$$u'(1) + ku(1) = 0 \quad \rightarrow \quad A + k(A + B) = 0 \tag{2}$$

Solving (1) and (2) for A and B yields

$$\begin{aligned} A &= -kB \\ -k^2 B &= 0. \end{aligned}$$

Hence, $A = 0$ and $B = 0$. Therefore, the solution in this case is the trivial one:

$$u(x) = 0.$$

Case $-$

Here we use the second ($-$) boundary condition.

$$u'(1) - ku(1) = 0 \quad \rightarrow \quad A - k(A + B) = 0 \tag{3}$$

Solving (1) and (3) for A and B yields

$$\begin{aligned} A &= -kB \\ B(k^2 - 2k) &= 0. \end{aligned}$$

Hence, $A = 0$ and $B = 0$, assuming $k \neq 0$ and $k \neq 2$. Therefore, the solution in this case is the trivial one:

$$u(x) = 0.$$

Case $k = 2$

In the case that $k = 2$, B is arbitrary and $A = -2B$. Therefore, we have a nontrivial solution:

$$u(x) = B(1 - 2x).$$